Exam August 15, 2017 in SF2852 Optimal Control.

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Allowed books: The formula sheet and β mathematics handbook, (or Tachenbuch Mathematischer Formeln).

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the optimization problem

$$\min \quad (x_3 - 1)^2 + \sum_{k=0}^{2} u_k^2 \quad \text{subj. to} \quad x_{k+1} = x_k + u_k, \; x_0 = 0$$

using dynamic programming. .... (10p)

2. In a factory with production rate $x(t)$ at time t it is possible to allocate a portion $u(t)$ of the production rate to reinvestment and $1 - u(t)$ to production of a storable goods. This means that $x(t)$ evolves according to

$$\dot{x}(t) = 0.1u(t)x(t)$$

where the constant 0.1 determines how much the reinvestment increases the production rate. Determine u such that the total amount of stored goods

$$\int_0^1 (1 - u(t))x(t)dt$$

is maximized subject to $0 \leq u(t) \leq 1$, for $t \in [0, 1]$. The initial production rate $x(0)$ is a known number. ....(10p)

3. This problem consists of two questions.

   (a) Determine which of the following partial differential equations $(a) - (d)$ corresponds to the following optimal control problem

$$\min_u x(1)^2 \quad \text{s.t} \quad \begin{cases} \dot{x} = 2u, & x(0) = x_0 \\ |u| \leq 1 \end{cases}$$

   (a) $-V_t = -2V_x \text{sign}(V_x)$, $V(1,x) = x^2$

   (b) $-V_t = -2V_x \text{sign}(V_x)$, $V(1,x) = 2x$

   (c) $-V_t = -2V_x$, $V(1,x) = x^2$

   (d) $-V_t = -\sin(V_x)(1 + 2V_x)$, $V(1,x) = x^2$

   .... (5p)

   (b) Consider the optimal control problem

$$J(x_0) = \min_u \int_0^1 f_{01}(x,u)dt + \int_1^2 f_{02}(x,u)dt$$
$$\text{s.t.} \quad \begin{cases} \dot{x} = f_1(x,u), & 0 \leq t \leq 1, \quad x(0) = x_0 \\ \dot{x} = f_2(x,u), & 1 \leq t \leq 2 \end{cases}$$

   Below are two attempts of solving the problem. They cannot both be correct (and may both be wrong). Find and explain the error(s) in the reasoning. Is any of the two attempts correct?

   Attempt 1:

$$\begin{aligned} J^*(x_0) = \ & \min_{u_1} \int_0^1 f_{01}(x_1,u_1)dt \quad + \quad \min_{x_1(1)} \min_u \int_1^2 f_{02}(x_2,u_2)dt \\ & s.t.\ \dot{x}_1 = f_1(x_1,u_1),\ x_1(0) = x_0 \qquad s.t.\ \dot{x}_2 = f_2(x_2,u_2),\ x_2(1) = x_1(1) \\ = \ & \min_{x_1(1)} J_1^*(x_0) + J_2^*(x_1(1)) \end{aligned}$$

   Attempt 2:

$$\begin{aligned} J^*(x_0) = \ & \min_u \{ \int_0^1 f_{01}(x,u)dt \quad + \quad \min_u \int_1^2 f_{02}(x,u)dt \ \} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 \qquad s.t.\ \dot{x} = f_2(x,u), \\ = \ & \min_u \left\{ \int_0^1 f_{01}(x,u)dt + J_2^*(x(1)) \right\} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 \end{aligned}$$

   where

$$\begin{aligned} J_k^*(x_0) = \ & \min_u \int_{k-1}^k f_{0k}(x_k,u)dt \\ & s.t.\ \dot{x}_k = f_k(x_k,u),\ x_k(k-1) = x_0 \end{aligned}$$

   .... (5p)

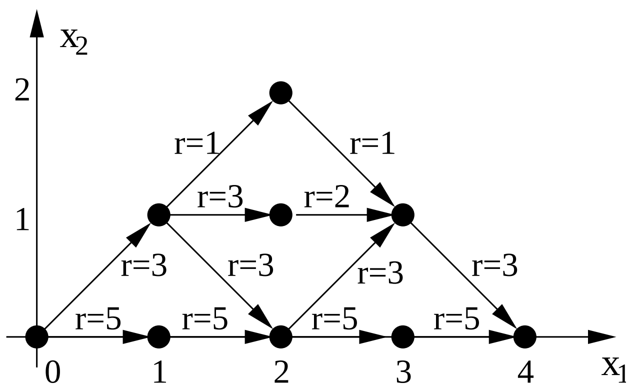

Figure 1: Graph with possible routes for the robot.

4. A robot is faced with the shortest path problem of going from node $(0, 0)$ to node $(4, 0)$ in the graph in Figure 1. We assume the simplest possible dynamics, i.e., the following kinematic equations.

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = u_2.$$

The cost of going along an arc from a node $n_1 = (x_{10}, x_{20})$ to a node $n_2 = (x_{11}, x_{21})$ is

$$c_{n_1,n_2}(r) = \min_{u_1,u_2} \quad r \int_0^1 (u_1^2 + u_2^2)dt \tag{1}$$
$$\text{subj. to} \quad \begin{cases} \dot{x}_1 = u_1, \ x_1(0) = x_{10}, \ x_1(1) = x_{11} \\ \dot{x}_2 = u_2, \ x_2(0) = x_{20}, \ x_2(1) = x_{21} \end{cases}$$

where the constant r determines the "energy" consumption per time unit.

(a) Use PMP to derive a formula for the optimal cost, $c_{n_1,n_2}(r)$, in (1). Notice that the optimal control moves the robot along the straight lines (arcs) between the nodes exactly as it is illustrated in the figure. .... (5p)

(b) Use (a) to compute the cost, $c_{n_1,n_2}(r)$, along each arc in the figure (the value of r is indicated above each arc and the coordinates for the nodes are given by the coordinate system). Then use dynamic programming to compute the shortest (minimum cost) path from node $(0, 0)$ to node $(4, 0)$. Indicate the optimal cost-to-go within parenthesis above each node and the optimal direction with an arrow at each node.
Remark: You don't have to formalize the dynamic programming calculations. .... (5p)

5. Consider the optimal control problem

$$\min T \qquad \begin{cases} \dot{x}_1 = 1 + u_1, & x_1(0) = x_1^0, \quad x_1(T) = 0 \\ \dot{x}_2 = u_2, & x_2(0) = x_2^0, \quad x_2(T) = 0 \\ u_1(t)^2 + u_2(t)^2 = 1 \end{cases}$$

We can interpret the optimal control problem as the problem of moving a point mass from a given position in the plane to zero in minimum time. The plane is tilted in x_1 direction, which gives the speed vector $(1 + u_1, u_2)$.

(*a*) Which are the controllable states? .... (3p)

(*b*) Use PMP to determine the optimal solution. .... (7p)

Good luck!

1. The dynamic programing recursion is

$$V(x,k+1) = \min_u \{u^2 + V(x+u,k)\}$$
$$V(x,3) = (x-1)^2$$

Simple calculations gives

$$u_0 = \frac{1}{4}(1-x_0) = \frac{1}{4}, \quad x_1 = \frac{1}{4}$$
$$u_1 = \frac{1}{3}(1-x_1) = \frac{1}{4}, \quad x_2 = \frac{1}{2}$$
$$u_2 = \frac{1}{2}(1-x_2) = \frac{1}{4}, \quad x_3 = \frac{3}{4}$$

2. The Hamiltonian is given by

$$H(x,\lambda) = (1-u)x + 0.1\lambda ux = x + (0.1\lambda - 1)ux.$$

From the problem formulation it is clear that $x(0) > 0$, hence $x(t) > 0$ for all $t \in [0,1]$. Therefore the control maximizing the Hamiltonian is

$$\mu(x,\lambda) = \begin{cases} 1 & \text{if } \lambda > 10 \\ 0 & \text{if } \lambda < 10 \\ [0,1] & \text{if } \lambda = 10. \end{cases}$$

Next, the dual dynamics is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -1 + (1 - 0.1\lambda)\mu(t,x) \geq \min(-1 - 0.1\lambda, -1),$$

and since there is no final constraint on $x(1)$ and no final cost, we have that $\lambda(1) = 0$.

Assume that $\lambda(t) \geq 10$ for some $t \in [0,1]$, then since $\lambda(1) = 0$ there must be a $\tilde{t}$ such that $\lambda(\tilde{t}) = 10$ and $\lambda(t) < 10$ for $t \in (\tilde{t}, 1)$. However, on $t \in (\tilde{t}, 1)$ we then have that $\dot{\lambda}(t) \geq \min(-1 - 0.1\lambda, -1) \geq -2$ which gives a contradiction since λ decreases from the value 10 to 0 in a time $1 - \tilde{t} \leq 1$. Therefore, $\lambda(t) < 10$ in the interval $[0,1]$ and the optimal control is $u^*(t) = 0$ for all $t \in [0,1]$.

3. (*a*) We have $H(x,u,\lambda) = 2\lambda u$. This gives $\tilde{\mu}(x,\lambda) = -\text{sign}(\lambda)$ and thus the HJBE becomes

$$\begin{cases} -V_t = H(x, \tilde{\mu}(x, V_x), V_x) \\ V(T,x) = \phi(x) \end{cases} \Leftrightarrow \begin{cases} -V_t = -2V_x \text{sign}(V_x) \\ V(1,x) = x^2 \end{cases}$$

Hence, alternative (*a*) is correct.

(*b*) The second attempt is using the dynamic programming equation correctly. In the first attempt the two time segments are treated independently, which is a violation of the dynamic programming equation and the principle of optimality.

4. (a) The Hamiltonian becomes $H(x, u, \lambda) = r(u_1^2 + u_2^2) + \lambda_1 u_1 + \lambda_2 u_2$. Pointwise minimization gives

$$\text{argmin}_u H(x, u, \lambda) \quad \Rightarrow \begin{cases} u_1 = -\frac{1}{2r}\lambda_1 \\ u_2 = -\frac{1}{2r}\lambda_2 \end{cases}$$

The adjoint equation becomes

$$\dot{\lambda} = -H_x(x, u, \lambda) \quad \Rightarrow \quad \begin{cases} \dot{\lambda}_1 = 0 \\ \dot{\lambda}_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda_1(t) = \lambda_1^0 & (constant) \\ \lambda_2(t) = \lambda_2^0 & (constant) \end{cases}$$

Hence, we have a constant control

$$\begin{cases} u_1(t) = u_1^0 & (constant) \\ u_2(t) = u_2^0 & (constant) \end{cases}$$

We use the boundary conditions to determine the constants

$$\begin{cases} x_{11} = u_1^0 + x_{10} \\ x_{21} = u_2^0 + x_{20} \end{cases} \quad \Rightarrow \quad \begin{cases} u_1^0 = x_{11} - x_{10} \\ u_2^0 = x_{21} - x_{20} \end{cases}$$

The optimal cost becomes $c_{n_1,n_2}(r) = r((x_{11}-x_{10})^2+(x_{21}-x_{20})^2)$.

(b) See Figure 2.

5. (*a*) The following states can be steered to zero $X = \{x \in \mathbf{R}^2 : x_1 < 0\} \cup \{(0, 0)\}$.

(*b*) The Hamiltonian is

$$H(x, u, \lambda) = 1 + \lambda_1(1 + u_1) + \lambda_2 u_2$$

Pointwise minimization gives

$$u^* = \mu(x, \lambda) = -\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

The adjoint equation shows that $(\lambda_1, \lambda_2) = (\lambda_1^0, \lambda_2^0)$ (constant). The state constraint gives

$$x_1(T) = (1 - \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}})T + x_1^0 = 0 \tag{2}$$

$$x_2(T) = -\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}T + x_2^0 = 0 \tag{3}$$

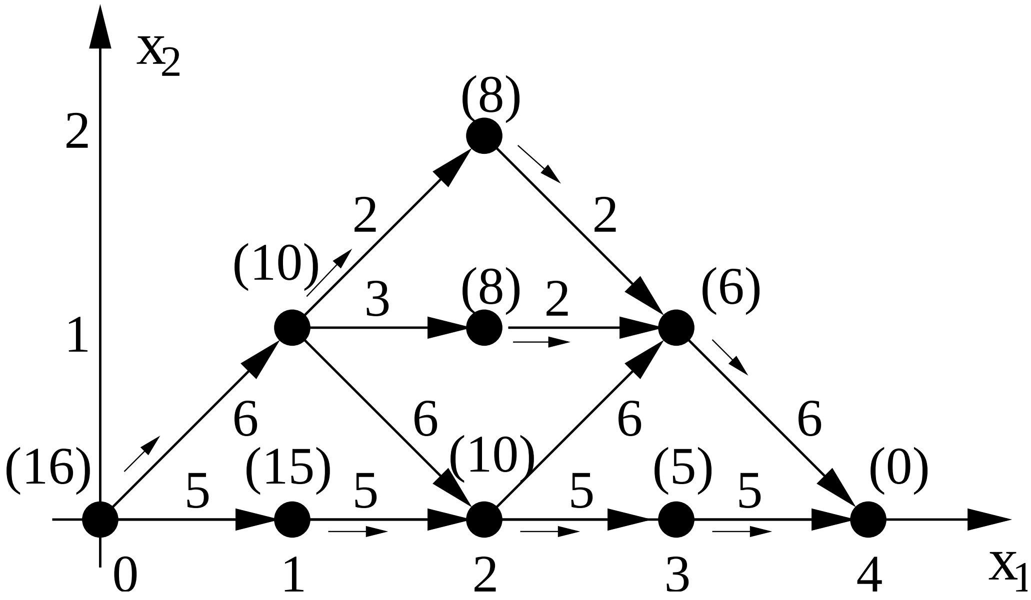

Figure 2: The cost is given above each arc. The optimal directions are given by arrows and the optimal cost-to-go are given within parentesis above each node.

If we square the two equations and add them together then we get

$$2(1 - \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}})T^2 = (x_1^0)^2 + (x_2^0)^2$$

Hence, from (2) we get

$$T = -\frac{(x_1^0)^2 + (x_2^0)^2}{2x_1^0}$$

and from (2) and (3)

$$u^* = -\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}} = \begin{bmatrix} -1 - \dfrac{x_1^0}{T} \\ -\dfrac{x_2^0}{T} \end{bmatrix}$$