Exam October 26, 2018 in SF2852 Optimal Control.

Examiner: Johan Karlsson, tel. 790 84 40.

Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Duration: 5 hours.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. In this problem we will determine an optimal production plan for a chemical plant. The problem is to purify 20 tons of substance using two available processes, A and B. The process A purifies all its input at a cost of $4u_A^2$, while process B purifies half of its input at the cost u_B^2. The processing is done over three stages as is illustrated in Figure 1. Here x_k denotes the amount of unpurified substance at stage k and u_k is the input to process B at stage k. By our assumption on process B we have $x_{k+1} = 0.5u_k$. In order to purify all substance we can only use process A in the last stage. The goal is to minimize the production cost.

   (*a*) Write down the optimal control problem. . . . . (4p)

   (*b*) Solve the optimal control problem using dynamic programming. (6p)

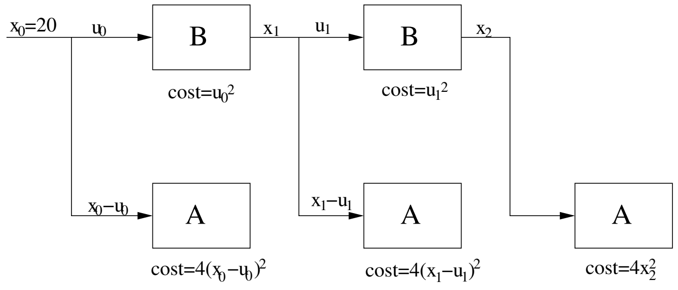

Figure 1: Three stage plant.

2. Each of the following four problems require brief motivation or brief calculations leading to the answer.

   (*a*) What is the optimal value for

$$J = \min t_f \quad \text{subj. to} \quad \begin{cases} \dot{x} = -x + u, & x(0) = 2,\ x(t_f) = 0 \\ u \in [-1, 1],\ t_f \geq 0 \end{cases}$$

   .... (2p)

   (*b*) What is the optimal value for

$$J = \min t_f \quad \text{subj. to} \quad \begin{cases} \dot{x} = -x + u, & x(0) = 0,\ x(t_f) = 2 \\ u \in [-1, 1],\ t_f \geq 0 \end{cases}$$

   .... (2p)

   (*c*) What is the optimal value for

$$J = \min \int_0^\infty (x^T Q x + u^T R u) dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = Ax + Bu, \\ x(0) = 0 \end{cases}$$

   where $Q \geq 0$ and $R > 0$.

   .... (2p)

   (*d*) Consider the optimal control problem

$$\min x_1(T) + \int_0^T f_0(x, u) dt \quad \text{subject to} \quad \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \\ x_1(T) + x_2(T) = 0 \\ x_1(T) - x_2(T) = 0 \end{cases}$$

   The state vector has n-variables ($x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$). Derive the boundary condition for the adjoint variable, i.e., the condition on $\lambda(T)$.

   .... (4p)

3. Consider the linear quadratic optimal control problem

$$\min \int_0^\infty (5x_1(t)^2 + u(t)^2)dt \quad \text{subject to} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases}$$

where $x(t) = [x_1(t), x_2(t)]^T$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a) Compute the optimal stabilizing feedback control and the corresponding optimal cost (if you use some theorem for this, verify that all conditions are satisfied). .... (7p)

(b) Compute the closed loop poles. .... (3p)

4. Consider the optimal control problem

$$\min T \qquad \begin{cases} \dot{x}_1 = 1 + u_1, & x_1(0) = x_1^0, \quad x_1(T) = 0 \\ \dot{x}_2 = u_2, & x_2(0) = x_2^0, \quad x_2(T) = 0 \\ u_1(t)^2 + u_2(t)^2 = 1 \end{cases}$$

We can interpret the optimal control problem as the problem of moving a point mass from a given position in the plane to zero in minimum time. The plane is tilted in x_1 direction, which gives the speed vector $(1 + u_1, u_2)$.

(a) Which are the controllable states (i.e., which are the initial states so that there is a feasible solution)? .... (3p)

(b) Use PMP to determine the optimal solution. .... (7p)

5. In this problem we will see how a runner can optimize his performance. The distance moved by the runner is determined by the equation

$$\dot{s}(t) = v_{\max} p(t) u(t), \qquad s(0) = 0$$

where $u(t) \in [0, 1]$ is the effort, $v_{\max}$ is the maximal speed and

$$p(t) = 1 - \int_0^t k e^{-k(t-s)} u(s) ds$$

is the degree of fitness ($k > 0$ and $v_{\max} > 0$ are constants). The goal is to find an optimal function $u(t)$ such that the distance run in T seconds is maximized.

(*a*) Formulate this as an optimal control problem. Let the states be $x_1(t) = s(t)$ and $x_2(t) = p(t)$.
.... (3p)

(*b*) Show that the optimal control is of bang-bang type and derive the switching function.
.... (2p)

(*c*) Show that the runner terminates the race running, i.e. $u(t) > 0$ on some interval $[T^*, T] \subset [0, T]$.
.... (2p)

(*d*) An optimal control is called singular if the switching function is zero on a nonzero time-interval. In this case we do not have a pure bang-bang solution since the control may take any value in $[0, 1]$ during the time interval when the switching function is zero. Show that it is possible to have a singular solution in our problem. Based on this, what type of solution do you expect to be optimal?
.... (3p)

Good luck!

Solutions

1. (*a*) The optimal control problem is

$$\min 4x_2^2 + \sum_{k=0}^{1}(4(x_k - u_k)^2 + u_k^2) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = 0.5u_k, & x_0 = 20 \\ 0 \le u_k \le x_k \end{cases}$$

(*b*) The dynamic programming equation becomes

$$J(k,x) = \min_{0\le u \le x} \left\{4(x-u)^2 + u^2 + J(k+1, 0.5u)\right\}$$
$$J(2,x) = 4x^2$$

We get

$$J(1,x) = \min_{0\le u\le x}\left\{4(x-u)^2 + u^2 + 4(0.5u)^2\right\} = \frac{4}{3}x^2 \quad \& \quad u_1 = \frac{2}{3}x_1$$
$$J(0,x) = \min_{0\le u\le x}\left\{4(x-u)^2 + u^2 + \frac{4}{3}(0.5u)^2\right\} \quad \Rightarrow \quad u_0 = \frac{3}{4}x_0$$

Hence we get following optimal inputs to process A and B:

	stage 0	stage 1	stage 2
Process A	5	2.5	2.5
Process B	15	5	0

2. (*a*) The answer is $t_f^* = \ln(3)$. The optimal control is clearly $u = -1$. This gives $x(t) = 3e^{-t} - 1$ which reaches 0 when $t = \ln(3)$.

(*b*) Note that for $x = 1$, then $\dot{x} = -1 + u \le 0$ since $u \in [-1, 1]$. Hence x can never cross from $x < 1$ to $x > 1$. Therefore $x = 2$ is not reachable from $x = 0$ and the cost is infinite.

(*c*) One solution is $x = 0, u = 0$ and the corresponding cost is 0. Since the cost $f_0(x,u) \ge 0$ this is the optimal solution.

(*d*) The boundary constraint becomes

$$\begin{bmatrix} \lambda_1(T) - 1 \\ \lambda_2(T) \\ \lambda_3(T) \\ \vdots \\ \lambda_n(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \nu_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \nu_2$$

where $\nu_1, \nu_2 \in \mathbb{R}$. It follows that $\lambda_1(T)$ and $\lambda_2(T)$ are free and the remaining adjoint variables are zero.

3. (*a*) We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, R = 1.$$

First note that (A, B) system is controllable since

$$[B, AB] = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

is full rank. Secondly, note that $C = \begin{pmatrix} \sqrt{5} & 0 \end{pmatrix}$ satisfy $Q = C^T C$ and (A, C) is observable since

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ \sqrt{5} & \sqrt{5} \end{pmatrix}$$

is full rank. Thus we can use Theorem 5 in the formula sheet. The ARE is $A^T P + PA + Q = PBR^{-1}B^T P$ and let

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix},$$

which gives the system of equations

$$2p_1 + 5 = (p_1 + p_2)^2, \tag{1}$$
$$p_1 + 2p_2 = (p_1 + p_2)(p_2 + p_3), \tag{2}$$
$$2p_2 + 2p_3 = (p_2 + p_3)^2. \tag{3}$$

Note that (3) only has solutions $p_2 + p_3 = 1 \pm 1$. Try the two cases.
First try $p_2 + p_3 = 2$. By (2) $p_1 = 0$ which cannot be a positive definite solution. Therefore we know that $p_2 + p_3 = 2$.
Using (2), we get $p_1 = -2p_2$. Plugging this into (1) gives $2p_1 + 5 = p_1^2/4$ which has the solutions $p_1 = 4 \pm 6$. The only solution that gives a positive definite P is $p_1 = 10$. This gives with the positive definite solution

$$P = \begin{pmatrix} 10 & -5 \\ -5 & 5 \end{pmatrix}.$$

and the optimal control

$$\hat{u} = -RB^T Px = -\begin{pmatrix} 5 & 0 \end{pmatrix} x = -5x_1.$$

The optimal cost is $J(x_0) = x_0^T P x_0 = (1, -1)P(1, -1)^T = 25$.

(*b*) The closed loop system is

$$\dot{x} = Ax - BR^{-1}B^TPx = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 5 & 0 \end{bmatrix}\right) x$$
$$= \begin{bmatrix} -4 & 1 \\ -5 & 1 \end{bmatrix} x = \hat{A}x.$$

The eigenvalues of $\hat{A}$ are $(-3 \pm \sqrt{5})/2$ which all have negative real parts, so the closed loop system is stable.

4. (*a*) The following states can be steered to zero $X = \{x \in \mathbf{R}^2 : x_1 < 0\} \cup \{(0,0)\}$.

(*b*) The Hamiltonian is

$$H(x, u, \lambda) = 1 + \lambda_1(1 + u_1) + \lambda_2 u_2$$

Pointwise minimization gives

$$u^* = \mu(x, \lambda) = -\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

The adjoint equation shows that $(\lambda_1, \lambda_2) = (\lambda_1^0, \lambda_2^0)$ (constant). The state constraint gives

$$x_1(T) = (1 - \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}})T + x_1^0 = 0 \qquad (4)$$
$$x_2(T) = -\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}T + x_2^0 = 0 \qquad (5)$$

If we square the two equations and add them together then we get

$$2(1 - \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}})T^2 = (x_1^0)^2 + (x_2^0)^2$$

Hence, from (4) we get

$$T = -\frac{(x_1^0)^2 + (x_2^0)^2}{2x_1^0}$$

and from (4) and (5)

$$u^* = -\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}} = \begin{bmatrix} -1 - \frac{x_1^0}{T} \\ -\frac{x_2^0}{T} \end{bmatrix}$$

5. (*a*) The optimization problem is

$$\min -x_1(T) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = v_{\max} x_2(t) u(t), & x_1(0) = 0 \\ \dot{x}_2(t) = -k x_2(t) + k(1 - u(t)), & x_2(0) = 1 \\ u \in [0, 1] \end{cases}$$

(*b*) The Hamiltonian is

$$H(x, u, \lambda) = \lambda_1 v_{\max} x_2 u + \lambda_2(-k x_2 + k(1 - u)).$$

From the pointwise minimization we get

$$\begin{aligned} u &= \text{argmin}_{u \in [0,1]} H(x, u, \lambda) = \text{argmin}_{u \in [0,1]} (\lambda_1 v_{\max} x_2 - \lambda_2 k) u \\ &= \begin{cases} 0, & \sigma < 0 \\ 1, & \sigma > 0 \end{cases} \end{aligned}$$

where $\sigma = \lambda_2 k - \lambda_1 v_{\max} x_2$.

(*c*) The adjoint equation becomes

$$\begin{aligned} \dot{\lambda}_1 &= 0, & \lambda_1(T) &= -1 \\ \dot{\lambda}_2 &= -v_{\max} u \lambda_1 + k \lambda_2, & \lambda_2(T) &= 0 \end{aligned}$$

Hence $\lambda_1(t) = -1$. Since $\lambda_2(T) = 0$ and $x_2(t) > 0$ for all $t > 0$, we get

$$\sigma(T) = \lambda_2(T) k - \lambda_1(T) v_{\max} x_2(T) = v_{\max} x_2(T) > 0.$$

Hence, $u(T) = 1$ and by continuity of $x_2(t)$ and $\lambda_2(t)$ it follows that there exists an interval $[T^*, T] \subset [0, T]$ on which $u(t) = 1$.

(*b*) We have

$$\dot{\sigma}(t) = -v_{\max} x_2 + k^2 \lambda_2 + v_{\max} k = k\sigma - 2k v_{\max} x_2 + v_{\max} k$$

Hence

$$\dot{\sigma}(t)_{|\sigma(t)=0} = 0$$

corresponds to $x_2(t) = 0.5$. This value of x_2 is an equilibrium point (i.e., $\dot{x}_2(t) = 0$) if $u = 0.5$.We have thus shown that there exists a singular solution corresponding to $u = 0.5$. One would expect that if the time interval is very long enough (T is large) then the optimal solution is on the form:

(a) first $u = 1$ until $x_2(t) = 0.5$
(b) then $u = 0.5$ until some point near the end
(c) then we let $u = 1$ again.

It is possible to prove that at most one switch could happen, which proves that the optimal solution must be of the form stated for large T.