Exam October 25, 2019 in SF2852 Optimal Control.

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Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Consider the optimal control problem

$$\min \sum_{k=0}^{\infty}(x_k^2 + 4u_k^2) \quad \text{subj.to} \quad x_{k+1} = 2(x_k + u_k),\ x_0 = \text{given}$$

   (*a*) Use the Bellman equation

$$V(x) = \min_u \{f_0(x,u) + V(f(x,u))\}$$

   to compute the optimal control and the optimal cost. ..... (8p)

   (*b*) Compute the eigenvalue of the closed loop system. .... (2p)

2. In this problem you will solve the following optimal control problem

$$\min \sum_{k=0}^{2}(x_k^2 + u_k^2) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + u_k, \quad x_0 = 0;\ x_3 = 2 \\ u_k \in U_k(x_k) = \{u : 0 \le x_k + u \le 2; u \text{ is an integer}\} \end{cases}$$

   (*a*) Formulate the dynamic programming algorithm for this problem. .... (2p)

   (*b*) Use the dynamic programming recursion to find the optimal solution. .... (8p)

3. Consider the optimal control problem

$$\min \int_0^{t_f} u(t)dt \quad \text{s.t.} \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), & x(0) = x_0 \quad x(t_f) = 0 \\ u \in [0, m] \end{cases} \tag{1}$$

(*a*) Suppose t_f is fixed. For what values of x_0 is it possible to find a solution to the above problem, i.e. for what values of x_0 can the constraints be satisfied?
.... (2p)

(*b*) Find the optimal control to (1) (for those x_0 you found in (*a*)). (5p)

(*c*) Let t_f be free, i.e. consider the optimal control problem

$$\min \int_0^{t_f} u(t)dt \quad \text{s.t.} \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), & x(0) = x_0 \quad x(t_f) = 0 \\ u \in [0, m]; \ t_f \geq 0 \end{cases}$$

Solve this optimal control problem for the case when $x_0 < 0$.
.... (3p)

4. Consider the following value function (cost-to-go function)

$$V(t_0, x_0) = \max_{u(t) \geq 0, t \in [t_0, T]} \int_{t_0}^{T} \sqrt{u(t)}dt$$
$$\text{s.t.} \begin{cases} \dot{x}(t) = \beta x(t) - u(t) \\ x(t) \geq 0, t \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

where $\beta > 0$. Verify that $V(t, x) = f(t)\sqrt{x}$, where

$$f(t) = \sqrt{\frac{e^{\beta(T-t)} - 1}{\beta}}$$

Comment: Note that the value function is only defined on $[0, T] \times R_+$*, where* $R_+ = (0, \infty)$ *(i.e.* $V : [0, T] \times R_+ \to R_+$*). The theorems presented in the course are also valid when the domain is restricted to such a set.*
.... (10p)

5. Consider the following optimization problem

$$\begin{aligned} \inf_{x(t),u(t),t\in[0,1]} & \int_0^1 tu(t)^2 dt && (2) \\ \text{subject to } & \dot{x}(t) = u(t) \\ & x(0) = 0, x(1) = 1 \end{aligned}$$

where $\inf(S)$ denotes the infimum of $S \subset \mathbf{R}$, i.e., the greatest lower bound on the numbers in S.

(*a*) Find the infimum in (2). .... (3p)

(*b*) Find a sequence of feasible control functions $u_k(t)$ such that the corresponding objective function in (2) converges to the infimum as $k \to \infty$. .... (7p)

Hint: Consider restricting the problem to a set of controllers that are zero in an interval in the beginning.

Good luck!

Solutions

1. (a) The Bellman equation becomes

$$V(x) = \min_u \{x^2 + 4u^2 + V(2x + 2u)\}$$

Let us try $V(x) = px^2$. We get

$$\begin{aligned} px^2 &= \min_u \{x^2 + 4u^2 + p(2x + 2u)^2\} \\ &= \min_u \{4(1 + p)(u + \frac{p}{1+p}x)^2 - \frac{4p^2}{1+p}x^2 + (1 + 4p)x^2\} \end{aligned}$$

This gives the optimal control

$$u = -\frac{p}{1+p}x$$

where p is the positive definite solution to the Riccati equation

$$\begin{aligned} & p = -\frac{4p^2}{1+p} + (1 + 4p) \\ \Leftrightarrow \quad & p^2 - 4p - 1 = 0 \\ \Leftrightarrow \quad & p = 2 \pm \sqrt{5} \end{aligned}$$

Hence, $p = 2 + \sqrt{5}$ and

$$\begin{aligned} u^* &= -\frac{2 + 1\sqrt{5}}{3 + \sqrt{5}}x \\ V(x_0) &= (2 + \sqrt{5})x_0^2 \end{aligned}$$

(b) The closed loop system is

$$x_{k+1} = \left(2 - \frac{4 + 2\sqrt{5}}{3 + \sqrt{5}}\right) x = \frac{2}{3 + \sqrt{5}}x_k,$$

and hence the eigenvalue is $\frac{2}{3+\sqrt{5}}$.

2. (a) The dynamic programming recursion can be stated

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} \left\{x_k^2 + u_k^2 + J_{k+1}(x_k + u_k)\right\}, \quad k = 0, 1, 2$$

$$J_3(x_3) = \begin{cases} 0, & x_3 = 2 \\ \infty, & \text{otherwise} \end{cases}$$

(b) For the second stage we have the following solution

$$J_2(x_2) = \min_{u_2 = 2 - x_2} \left\{x_2^2 + u_2^2\right\} = 2x_2^2 - 4x_2 + 4$$

Hence, the solution is given according to the following table

x_2	$J_2(x_2)$	$\mu(2, x_2)$
0	4	2
1	2	1
2	4	0

In the first stage we get

$$J_1(x_1) = \min_{-x_1 \le u_1 \le 2-x_1} \underbrace{\{x_1^2 + u_1^2 + J_2(x_1 + u_1)\}}_{J_1(x_1,u_1)}$$

The cost $J_1(x_1, u_1)$ for all feasible pairs (u_1, x_1) are given in the following table.

x_1	$u_1 = -2$	-1	0	1	2
0			4	3	8
1		6	3	6	
2	12	7	8		

which implies the follow optimal solution

x_1	$J_1(x_1)$	$\mu(1, x_1)$
0	3	1
1	3	0
2	7	-1

In the initial stage we have the Bellman recursion

$$J_0(x_0) = \min_{-x_0 \le u_0 \le 2-x_0} \underbrace{\{x_0^2 + u_0^2 + J_1(x_0 + u_0)\}}_{J_0(x_0,u_0)}$$

The cost $J_0(x_0)$ for all feasible pairs (x_0, u_0) are given in the following table

x_0	$u_0 = -2$	-1	0	1	2
0			3	4	11
1		5	4	9	
2	11	8	11		

and thus

x_0	$J_0(x_0)$	$\mu(0, x_0)$
0	3	0
1	4	0
2	8	-1

The optimal control is thus

$$u_0^* = 0,\ u_1^* = 1,\ u_2^* = 1$$

and the corresponding optimal state trajectory is

$$x_0^* = 0,\ x_1^* = 0,\ x_2^* = 1,\ x_3^* = 2$$

3. (*a*) The solution to the differential equation is

$$x(t) = e^{-t}x_0 + \int_0^t e^{-(t-s)}u(s)ds$$

Since $u(t) \geq 0$ it follows that $x(t) \geq e^{-t}x_0$, which never can become zero in finite time unless $x_0 \leq 0$. Suppose $x_0 < 0$. It is possible to bring the state back to zero in time t_f if

$$e^{-t_f}x_0 + (1 - e^{-t_f})m \geq 0$$

i.e., if $x_0 \geq -(e^{t_f} - 1)m$.
Hence, the constraint can be satisfied if $x_0 \in [-(e^{t_f} - 1)m, 0]$.

(*b*) The Hamiltonian function is $H(x, u, \lambda) = u + \lambda(-x + u)$. Pointwise minization gives

$$\begin{aligned} u^* = \bar{\mu}(x,\lambda) &= \text{argmin}_{u\in[0,m]} H(x,u,\lambda) = \text{argmin}_{u\in[0,m]}(1+\lambda)u \\ &= \begin{cases} 0, & \lambda > -1 \\ m, & \lambda < -1 \end{cases} \end{aligned}$$

The adjoint differential equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \lambda$$

which has the solution $\lambda(t) = e^t\lambda_0$, for some $\lambda_0 \in R$. Hence, the optimal control is

$$u^*(t) = \begin{cases} 0, & e^t\lambda_0 > -1 \\ m, & e^t\lambda_0 < -1 \end{cases}$$

Hence, there are three cases

(a) $\lambda_0 \geq 0 \Rightarrow u^*(t) = 0,\ t \in [0, t_f]$
(b) $\lambda_0 \in (-1, 0) \Rightarrow u^*(t)$ will switch from $u^*(t) = 0$ to $u^*(t) = m$ at $t = -\ln(|\lambda_0|)$
(c) $\lambda_0 < -1 \Rightarrow u^*(t) = m,\ t \in [0, t_f]$

Hence, the possible control sequences are $\{0\}$, $\{0, 1\}$, and $\{1\}$, i.e. the optimal control is

$$u^*(t) = \begin{cases} 0, & 0 \leq t \leq t^* \\ m, & t^* \leq t \leq t_f \end{cases}$$

where $t^* \in [0, t_f]$ must be determined. One way to determine t^* is to consider the closed form solution

$$x^*(t_f) = e^{-t_f}x_0 + \int_{t^*}^{t_f} e^{-(t_f-s)}u^*(s)ds = e^{-t_f}x_0 + (1 - e^{-(t_f-t^*)})m = 0$$

Hence, we get

$$t^* = t_f + \ln\left(\frac{m + e^{-t_f}x_0}{m}\right)$$

which is between $[0, t_f]$ when $x_0 \in [-(e^{t_f} - 1)m, 0]$.

(c) Let us investigate the condition

$$H(x^*(t), u^*(t), \lambda(t)) = 0, \ t \in [0, t_f^*]$$

We already now from above that the optimal control is of bang-bang type and there is at most one switch from $u^*(t) = 0$ to $u^*(t) = m$. Suppose $u^*(t) = 0$ on $t \in [0, t^*]$. Then

$$H(x^*(t), u^*(t), \lambda(t)) = \lambda(t)x^*(t) = e^t\lambda_0 e^{-t}x_0 = \lambda_0 x_0 = 0, \ t \in [0, t^*]$$

However, we know from above that $\lambda_0 < 0$ if there is a switch and $x_0 < 0$ by assumption. This contradicts the assumption that there is a switch and it follows that the optimal control is $u^*(t) = 0$, for all $t \geq 0$. Note that this implies $t_f^* = \infty$, i.e. it takes infinitely long time to do the transfer but the cost is zero.

4. We use the verification theorem, i.e., we need to verify that

$$-\frac{\partial V}{\partial t}(t, x) = \max\left\{\sqrt{u} + \frac{\partial V}{\partial x}(t, x)^T(\beta x - u)\right\}$$

for some $V : [0, T] \times (0, \infty) \to R$ With $V(t, x) = f(t)\sqrt{x}$, we get

$$\begin{aligned}
-\dot{f}(t)\sqrt{x} &= \max\left\{\sqrt{u} + \frac{f(t)}{2\sqrt{x}}(\beta x - u)\right\} \\
&= \frac{\sqrt{x}}{f(t)} + \frac{f(t)}{2\sqrt{x}}(\beta x - \frac{x}{f(t)^2}
\end{aligned}$$

where we used that $u = \sqrt{x}/f(t)$ is the maximizing control. This equation is satisfied if

$$-f(t)\dot{f}(t) = \frac{1}{2}(1 + \beta f(t)^2)$$

which implies

$$f(t)^2 = \frac{e^{\beta(T-t)} - 1}{\beta}.$$

Finally notice that $V(T, x) = f(T)\sqrt{x} = 0$, since $f(T) = 0$.

5. For a given $0 < \epsilon < 1$, consider the optimal controller u_ϵ given the additional constraint that it must be zero in the interval $[0, \epsilon]$. The corresponding restricted optimization problem is

$$\begin{aligned}\inf_{x(t),u(t),t\in[\epsilon,1]} & \int_\epsilon^1 tu(t)^2 dt \\ \text{subject to } & \dot{x}(t) = u(t) \\ & x(\epsilon) = 0, \quad x(1) = 1.\end{aligned}$$

The Hamiltonian is given by $H(t, x, u, \lambda) = tu^2 + \lambda u$, and the minimizing u is given by

$$\mu(t, x, \lambda) = -\frac{\lambda}{2t}.$$

The adjoint dynamics are

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0,$$

hence λ is constant. The optimal control for the restricted problem is thus $u_\epsilon(t) = -\frac{\lambda}{2t}$ where λ is selected so that $x(1) = 1$ is satisfied. This can be determined by

$$1 = x(1) = \int_\epsilon^1 u(t)dt = \int_\epsilon^1 -\frac{\lambda}{2t}dt = \frac{\lambda}{2}\log(\epsilon),$$

and thus $\lambda = 2/\log(\epsilon)$ and

$$u_\epsilon(t) = \begin{cases} 0 & t \in [0, \epsilon) \\ -\frac{1}{t\log(\epsilon)} & t \in [\epsilon, 1] \end{cases}$$

is the optimal controller subject to the restricted that it is zero on $[0, \epsilon]$. The cost corresponding to this controller is

$$\int_\epsilon^1 tu(t)^2 dt = \int_\epsilon^1 \frac{\lambda^2}{4t}dt = -\frac{\lambda^2}{4}\log(\epsilon) = -\frac{1}{\log(\epsilon)}.$$

Note that $-\frac{1}{\log(\epsilon)} \to 0$ as $\epsilon \to 0$, hence by selecting a sequence of controllers as u_ϵ and letting $\epsilon \to 0$ we see that the infimum is less or equal to 0. Further, note that the cost is always non-negative, hence 0 is the greatest lower bound to the cost.