Solutions to the exam in SF2862, June 2009

Exercise 1.
This is a deterministic periodic-review inventory model. Let
$n =$ the number of considered weeks, i.e. $n = 4$ in this exercise, and
$r_i =$ the demand at week i, i.e. $r_1 = r_2 = r_3 = r_4 = 100$ in this exercise.

The total cost consists of three parts: The ordering costs for orders, the holding costs, and the metal cost. But the latter is $1000 \times (r_1 + r_2 + r_3 + r_4)$ for all feasible order plans, so this unavoidable metal cost may simply be ignored when searching for an optimal order plan.

Let $C_i^{(j)}$ = the minimal remaining (ordering+holding) costs from week i, given that the inventory is empty at the end of week $i-1$ and then filled in such a way that the next time it will be empty is by the end of week j. Then $C_i^{(j)} = K + h\cdot(r_{i+1} + 2r_{i+2} + \cdots + (j-i)r_j) + C_{j+1}$.

Further, let C_i = the minimal remaining (ordering+holding) costs from week i, given that the inventory is empty at the end of week $i-1$. Then $C_i = \min\{\, C_i^{(i)}, C_i^{(i+1)}, \ldots, C_i^{(n)}\,\}$.

(a). Here, $K = 700$ and $h = 2$. We then get that

$C_4 = C_4^{(4)} = 700.$

$C_3^{(4)} = 700 + 200 = 900.$
$C_3^{(3)} = 700 + C_4 = 1400.$
$C_3 = \min\{\, C_3^{(3)}, C_3^{(4)}\,\} = 900.$

$C_2^{(4)} = 700 + 200 + 400 = 1300.$
$C_2^{(3)} = 700 + 200 + C_4 = 1600.$
$C_2^{(2)} = 700 + C_3 = 1600.$
$C_2 = \min\{\, C_2^{(2)}, C_2^{(3)}, C_2^{(4)}\,\} = 1300.$ $C_1^{(4)} = 700 + 200 + 400 + 600 = 1900.$
$C_1^{(3)} = 700 + 200 + 400 + C_4 = 2000.$
$C_1^{(2)} = 700 + 200 + C_3 = 1800.$
$C_1^{(1)} = 700 + C_2 = 2000.$
$C_1 = \min\{\, C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}\,\} = 1800.$

The optimal plan is to order 200 kg before the first week and 200 kg before the third week.

(b). Here, $K = 700 + c > 700$ and $h = 2$. We then get that

$C_4 = C_4^{(4)} = 700 + c.$

$C_3^{(4)} = 700 + c + 200 = 900 + c.$
$C_3^{(3)} = 700 + c + C_4 = 1400 + 2c.$
$C_3 = \min\{\, C_3^{(3)}, C_3^{(4)}\,\} = 900 + c.$

$C_2^{(4)} = 700 + c + 200 + 400 = 1300 + c.$
$C_2^{(3)} = 700 + c + 200 + C_4 = 1600 + 2c.$
$C_2^{(2)} = 700 + c + C_3 = 1600 + 2c.$
$C_2 = \min\{\, C_2^{(2)}, C_2^{(3)}, C_2^{(4)}\,\} = 1300 + c.$

$C_1^{(4)} = 700 + c + 200 + 400 + 600 = 1900 + c.$
$C_1^{(3)} = 700 + c + 200 + 400 + C_4 = 2000 + 2c.$
$C_1^{(2)} = 700 + c + 200 + C_3 = 1800 + 2c.$
$C_1^{(1)} = 700 + c + C_2 = 2000 + 2c.$
$C_1 = \min\{ C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)} \}.$

If $0 < c < 100$ then $C_1 = C_1^{(2)} = 1800 + 2c$, and then the optimal plan is
to order 200 kg before the first week and 200 kg before the third week.
If $c > 100$ then $C_1 = C_1^{(4)} = 1900 + c$, and then the optimal plan is
to order 400 kg before the first week.

Exercise 3.
The solution of this exercise is best illustrated by drawing a decision tree, but since we are reluctant to do this in latex, we present the solution in a much more boring way.

Let **H1** be the decision of making a hard first serve.
Let **L1** be the decision of making a lob first serve.
Let **H2** be the decision of making a hard second serve.
Let **L2** be the decision of making a lob second serve.
Let **IN** be the event that the serve is in bounds.
Let **OUT** be the event that the serve is not in bounds.

H1
A hard first serve is in bounds with prob p, and out of bounds with prob $1-p$.

H1 - IN
Here, MM wins the point with prob 3/4 and loses the point with prob 1/4.
The expected cost at this node is thus $(3/4) \cdot (-1) + (1/4) \cdot (+1) = -1/2$.

H1 - OUT
There are two alternatives for the second serve: hard or lob.

H1 - OUT - H2
A hard second serve is in bounds with prob p, and out of bounds with prob $1-p$.

H1 - OUT - H2 - IN
Here, MM wins the point with prob 3/4 and loses the point with prob 1/4.
The expected cost at this node is thus $(3/4) \cdot (-1) + (1/4) \cdot (+1) = -1/2$.

H1 - OUT - H2 - OUT
Here, MM loses the point. The expected cost at this node is thus +1.

H1 - OUT - H2
The expected cost at this node is thus $p \cdot (-1/2) + (1-p) \cdot (+1) = 1-3p/2$.

H1 - OUT - L2
A lob second serve is in bounds with prob q, and out of bounds with prob $1 - q$.

H1 - OUT - L2 - IN
Here, MM wins the point with prob 1/2 and loses the point with prob 1/2.
The expected cost at this node is thus $(1/2) \cdot (-1) + (1/2) \cdot (+1) = 0$.

H1 - OUT - L2 - OUT
Here, MM loses the point. The expected cost at this node is thus $+1$.

H1 - OUT - L2
The expected cost at this node is thus $q \cdot 0 + (1-q) \cdot (+1) = 1-q$.

H1 - OUT
The minimal expected cost at this node is thus $\min\{\ 1-3p/2,\ 1-q\ \}$.

H1
The minimal expected cost at this node is thus $p \cdot (-1/2) + (1-p) \cdot \min\{\ 1-3p/2,\ 1-q\ \}$.

L1
A lob first serve is in bounds with prob q, and out of bounds with prob $1-q$.

L1 - IN
Here, MM wins the point with prob 1/2 and loses the point with prob 1/2.
The expected cost at this node is thus $(1/2) \cdot (-1) + (1/2) \cdot (+1) = 0$.

L1 - OUT
There are two alternatives for the second serve: hard or lob.

L1 - OUT - H2
A hard second serve is in bounds with prob p, and out of bounds with prob $1-p$.

L1 - OUT - H2 - IN
Here, MM wins the point with prob 3/4 and loses the point with prob 1/4.
The expected cost at this node is thus $(3/4) \cdot (-1) + (1/4) \cdot (+1) = -1/2$.

L1 - OUT - H2 - OUT
Here, MM loses the point. The expected cost at this node is thus $+1$.

L1 - OUT - H2
The expected cost at this node is thus $p \cdot (-1/2) + (1-p) \cdot (+1) = 1-3p/2$.

L1 - OUT - L2
A lob second serve is in bounds with prob q, and out of bounds with prob $1 - q$.

L1 - OUT - L2 - IN
Here, MM wins the point with prob 1/2 and loses the point with prob 1/2.
The expected cost at this node is thus $(1/2) \cdot (-1) + (1/2) \cdot (+1) = 0$.

L1 - OUT - L2 - OUT
Here, MM loses the point. The expected cost at this node is thus $+1$.

L1 - OUT - L2
The expected cost at this node is thus $q \cdot 0 + (1-q) \cdot (+1) = 1-q$.

L1 - OUT
The minimal expected cost at this node is thus $\min\{\ 1-3p/2,\ 1-q\ \}$.

L1
The minimal expected cost at this node is thus $q \cdot 0 + (1-q) \cdot \min\{\ 1-3p/2,\ 1-q\ \}$.

From these calculations, we get that the minimal expected cost before making the first serve is given by

$\min\{ -p/2 + (1-p) \cdot \min\{ 1-3p/2,\ 1-q \},\ (1-q) \cdot \min\{ 1-3p/2,\ 1-q \} \}.$

Alternatively, this minimal expected cost can be written

$\min\{ \mathrm{F}_{\mathrm{HH}}(p,q), \mathrm{F}_{\mathrm{HL}}(p,q), \mathrm{F}_{\mathrm{LH}}(p,q), \mathrm{F}_{\mathrm{LL}}(p,q) \}$, where

$\mathrm{F}_{\mathrm{HH}}(p,q) = -p/2 + (1-p)(1-3p/2),$
$\mathrm{F}_{\mathrm{HL}}(p,q) = -p/2 + (1-p)(1-q),$
$\mathrm{F}_{\mathrm{LH}}(p,q) = (1-q)(1-3p/2),$
$\mathrm{F}_{\mathrm{LL}}(p,q) = (1-q)^2.$

(a). If $p = 1/2$ and $q = 7/8$ then

$\mathrm{F}_{\mathrm{HH}}(p,q) = -1/8,$
$\mathrm{F}_{\mathrm{HL}}(p,q) = -3/16,$
$\mathrm{F}_{\mathrm{LH}}(p,q) = 1/32,$
$\mathrm{F}_{\mathrm{LL}}(p,q) = 1/64,$

which shows that the optimal strategy is a hard first serve and a lob second serve.

(b). We have that

$\mathrm{F}_{\mathrm{LH}}(p,q) - \mathrm{F}_{\mathrm{HL}}(p,q) = (1-q)(1-3p/2) + p/2 - (1-p)(1-q) = pq/2 > 0,$

which shows that the strategy “L1–H2” is always inferior to the strategy “H1–L2”.

Exercise 4.

The arrival rates to the two facilities are obtained from the system

$\lambda_1 = 9\,p + 0.2\lambda_2$ and $\lambda_2 = 9\,(1{-}p) + 0.5\lambda_1$,

which gives that $\lambda_1 = 2 + 8p$ and $\lambda_2 = 10 - 5p$.

We know that both F_1 and F_2 are $M/M/1$ with $\mu_1 = \mu_2 = 10$,
so that $\rho_1 = \lambda_1/\mu_1 = 0.2 + 0.8p$ and $\rho_2 = \lambda_2/\mu_2 = 1 - 0.5p$.

(a) The system can be in steady state if and only if both $\rho_1 < 1$ and $\rho_2 < 1$ (with strict inequalities), which is equivalent to that $0 < p < 1$ (with strict inequalities).
In particular, the system can not be in steady state if $p = 0$ or $p = 1$.

(b) Assume that $0 < p < 1$. Then

$$L_1 = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{2+8p}{8-8p} = -1 + \frac{10}{8-8p} \quad \text{and} \quad L_2 = \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{10-5p}{5p} = -1 + \frac{10}{5p}\,,$$

so that the average number of customers in the system is

$$L_1 + L_2 = -2 + \frac{10}{8-8p} + \frac{10}{5p} = -2 + \frac{1.25}{1{-}p} + \frac{2}{p}\,.$$

This number should be minimized with respect to $p \in (0,1)$.

Let $f(p) = -2 + \dfrac{1.25}{1{-}p} + \dfrac{2}{p}$. Then $f'(p) = \dfrac{1.25}{(1{-}p)^2} - \dfrac{2}{p^2}$ and $f''(p) = \dfrac{2.5}{(1{-}p)^3} + \dfrac{4}{p^3}\,.$

Since $f''(p) > 0$ for all $p \in (0,1)$, f is strictly convex on this interval, so we search for a solution to $f'(p) = 0$, which after some simple calculations gives that the unique optimal p is

$$p = \frac{\sqrt{2}}{\sqrt{2} + \sqrt{1.25}} = \frac{2}{2 + \sqrt{2.5}} \approx \frac{2}{2 + 1.6} = \frac{5}{9}\,.$$

(c) Assume again that $0 < p < 1$. Then the steady state probability that facilty F_1 is empty is $1 - \rho_1 = 0.8(1{-}p)$ and the corresponding probability for F_2 is $1 - \rho_2 = 0.5p$. The steady state probability that the whole system is empty is then given by $(1{-}\rho_1)(1{-}\rho_2) = 0.4p(1{-}p)$, which should be maximized. Simple calculations shows that the unique optimal p is $p = 0.5$, in which case the steady state probability for an empty system is 0.1.

(d) Let V_j be the expected time for a customer who arrives to facility F_j to go through that facility once. Then $V_j = \dfrac{1}{\mu_j - \lambda_j}$, so that $V_1 = \dfrac{1}{8-8p}$ and $V_2 = \dfrac{1}{5p}\,.$

Let W_j be the expected remaining time in the system for a customer who arrives to facility F_j. Then $W_1 = V_1 + 0.5W_2$ and $W_2 = V_2 + 0.2W_1$, which gives that

$$W_1 = \frac{10/9}{8-8p} + \frac{5/9}{5p} \quad \text{and} \quad W_2 = \frac{2/9}{8-8p} + \frac{10/9}{5p}\,.$$

A randomly chosen new customer will with probablity p first go to F_1, and with probablity $1{-}p$ first go to F_2. Therefore, the expected total time in the system for a new customer is

$$pW_1 + (1{-}p)W_2 = \frac{1}{9}\left(\frac{2+8p}{8-8p} + \frac{10-5p}{5p}\right) = \frac{L_1 + L_2}{9}.$$

The optimal p is thus the same as in **(b)** above.

Exercise 5.

Assume that the false coin is known to be among n specific coins.
If Hook puts k coins in each bowl, where $k \geq 1$ and $2k \leq n$, then one of the following two things will happen.

The two bowls contain equal weights, in which case the false coin is among the left out $n{-}2k$ coins. After this, the minimal numbers of additional trials (in worst case) is $V(n{-}2k)$.

The bowls contain different weights, in which case the false coin is among the k coins in the lightest bowl. After this, the minimal numbers of additional trials (in worst case) is $V(k)$.

So after the trial with k coins in each bowl, the minimal numbers of additional trials will (in worst case) be the largest of the two numbers $V(n{-}2k)$ and $V(k)$, i.e. $\max\{V(n{-}2k),\, V(k)\}$.

Note that if n is even and $k = n/2$, then the bowls cannot contain equal weights, so then $\max\{V(n-2k),\, V(k)\}$ ought to be replaced simply by $V(k)$. But this replacement is not needed if we define $V(0) = 0$.

The above discussion leads to the recursive equation:
$V(n) = 1 + \min_k \{ \max\{V(n{-}2k),\, V(k)\} \}$,
where k must satisfy $1 \leq k \leq \dfrac{n}{2}$, and where $V(0) = V(1) = 0$.

$V(2) = 1 + \min_k \{ \max\{V(2{-}2k),\, V(k)\} \} = 1 + \{ \max\{V(0),\, V(1)\} \} = 1.$ Optimal $k = 1$.

$V(3) = 1 + \min_k \{ \max\{V(3{-}2k),\, V(k)\} \} = 1 + \{ \max\{V(1),\, V(1)\} \} = 1.$ Optimal $k = 1$.

$V(4) = 1 + \min_k \{ \max\{V(4{-}2k),\, V(k)\} \} = 1 + \min \{ \max\{V(2),\, V(1)\},\, \max\{V(0),\, V(2)\} \} =$
$= 1 + \min \{ \max\{1,\, 0\},\, \max\{0,\, 1\} \} = 1 + 1 = 2.$ Optimal $k = 1$ or 2.

$V(5) = 1 + \min_k \{ \max\{V(5{-}2k),\, V(k)\} \} = 1 + \min \{ \max\{V(3),\, V(1)\},\, \max\{V(1),\, V(2)\} \} =$
$= 1 + \min \{ \max\{1,\, 0\},\, \max\{0,\, 1\} \} = 1 + 1 = 2.$ Optimal $k = 1$ or 2.

$V(6) = 1 + \min_k \{ \max\{V(6{-}2k),\, V(k)\} \} =$
$1 + \min \{ \max\{V(4),\, V(1)\},\, \max\{V(2),\, V(2)\},\, \max\{V(0),\, V(3)\} \} =$
$1 + \min \{ \max\{2,\, 0\},\, \max\{1,\, 1\},\, \max\{0,\, 1\} \} = 1 + 1 = 2.$ Optimal $k = 2$ or 3.

$V(7) = 1 + \min_k \{ \max\{V(7{-}2k),\, V(k)\} \} =$
$1 + \min \{ \max\{V(5),\, V(1)\},\, \max\{V(3),\, V(2)\},\, \max\{V(1),\, V(3)\} \} =$
$1 + \min \{ \max\{2,\, 0\},\, \max\{1,\, 1\},\, \max\{0,\, 1\} \} = 1 + 1 = 2.$ Optimal $k = 2$ or 3.

$V(8) = 1 + \min_k \{ \max\{V(8{-}2k),\, V(k)\} \} =$
$1 + \min \{ \max\{V(6),\, V(1)\},\, \max\{V(4),\, V(2)\},\, \max\{V(2),\, V(3)\}\ \max\{V(0),\, V(4)\} \} =$
$1 + \min \{ \max\{2,\, 0\},\, \max\{2,\, 1\},\, \max\{1,\, 1\},\, \max\{0,\, 2\} \} = 1 + 1 = 2.$ Optimal $k = 3$.

$V(9) = 1 + \min_k \{ \max\{V(9{-}2k),\, V(k)\} \} =$
$1 + \min \{ \max\{V(7),\, V(1)\},\, \max\{V(5),\, V(2)\},\, \max\{V(3),\, V(3)\}\ \max\{V(1),\, V(4)\} \} =$
$1 + \min \{ \max\{2,\, 0\},\, \max\{2,\, 1\},\, \max\{1,\, 1\},\, \max\{0,\, 2\} \} = 1 + 1 = 2.$ Optimal $k = 3$.

So the optimal strategy for Captain Hook is to first put 3 coins in each bowl and let 3 coins be left out. After the first balancing, there will be just 3 coins to choose between. Then one more balancing is needed, with one coin in each bowl and one left out.