Solutions to the exam in SF2862, March 2009

Exercise 1.

Let S be the number of overbookings made by Happy.
If $\xi > S$ then there are $\xi - S$ empty seats.
If $\xi < S$ then there are $S - \xi$ bumped passengers.
Thus, the expected cost is $C(S) = p\,\mathrm{E}(\xi - S)^+ + h\,\mathrm{E}(S - \xi)^+$,
where $p = 300$ and $h = 600$.

Let $\Delta C(S) = C(S+1) - C(S) = p\,(F_\xi(S) - 1) + hF_\xi(S) = 900F_\xi(S) - 300$.

Then $\Delta C(S)$ is nondecreasing in S and $C(S) = C(0) + \Delta C(0) + \cdots + \Delta C(S-1)$.

Therefore, $S^\star$ minimizes $C(S)$ if and only if $\Delta C(0) \leq \cdots \leq \Delta C(S^\star - 1) \leq 0$,
while $0 \leq \Delta C(S^\star) \leq \Delta C(S^\star + 1) \leq \cdots\cdots$, which is equivalent to that

$$F_\xi(S^\star - 1) \leq \frac{300}{900} \leq F_\xi(S^\star).$$

For the given discrete random variable ξ, it holds that
$F_\xi(0) = 0$,
$F_\xi(1) = 1/25$,
$F_\xi(2) = (1+2)/25 = 3/25$,
$F_\xi(3) = (1+2+3)/25 = 6/25 < 300/900$,
$F_\xi(4) = (1+2+3+4)/25 = 10/25 > 300/900$.

Thus, $S^\star = 4$.

If ξ is assumed to be a continuous random variable then
$C'(S) = p\,(F_\xi(S) - 1) + hF_\xi(S) = 900F_\xi(S) - 300$,
while $C''(S) = 900f_\xi(S) \geq 0$ for all S, so that $C(S)$ is a convex function.
Therefore, $S^\star$ minimizes $C(S)$ if and only if $F_\xi(S^\star) = \dfrac{300}{900}$.
For the given continuous random variable ξ, it holds that
$F_\xi(S) = \dfrac{S^2}{50}$ for $S \in [0,\,5]$, while $F_\xi(S) = 1 - \dfrac{(10-S)^2}{50}$ for $S \in [5,\,10]$.

Thus, $S^\star = \sqrt{50/3} \approx \sqrt{16.7}$, which rounded to nearest integer again gives that the number of overbookings should be 4.

Exercise 2.

The arrival rates to the three facilities are obtained from the system

$\lambda_1 = 20,$
$\lambda_2 = 0.4 \cdot \lambda_1 + (2/3) \cdot \lambda_3,$
$\lambda_3 = 0.3 \cdot \lambda_1 + (3/4) \cdot \lambda_2,$

which gives that $\lambda_1 = 20,\ \lambda_2 = 24,\ \lambda_3 = 24.$

We know that F1 is $M/M/2$ with $\mu_1 = 20$, so that $\rho_1 = \lambda_1/(2\mu_1) = 0.5$,
while F2 and F3 both are $M/M/1$ with $\mu_2 = \mu_3 = 30$, so that $\rho_2 = \rho_3 = 0.8$.
Therefore,

$$L_1 = \frac{2\rho_1}{1-\rho_1^2} = \frac{1}{1-0.5^2} = \frac{4}{3},\quad L_2 = \frac{\rho_2}{1-\rho_2} = 4 \text{ and } L_3 = \frac{\rho_3}{1-\rho_3} = 4.$$

So the average number of customers in the system is $L_1 + L_2 + L_3 = 9\frac{1}{3}$.

Let N_j be the number of customers at facility j (at a randomly chosen point in time).

Then

$$P(N_1\!=\!0) = \frac{1-\rho_1}{1+\rho_1} = \frac{1}{3},\quad P(N_1\!=\!1) = 2\rho_1\frac{1-\rho_1}{1+\rho_1} = \frac{1}{3},$$

$$P(N_2\!=\!0) = 1-\rho_2 = 0.2,\quad P(N_2\!=\!1) = \rho_2(1-\rho_2) = 0.16,$$

$$P(N_3\!=\!0) = 1-\rho_3 = 0.2,\quad P(N_3\!=\!1) = \rho_3(1-\rho_3) = 0.16,$$

The probablity that there is exactly one customer in the system is then given by

$$P(N_1\!=\!1)P(N_2\!=\!0)P(N_3\!=\!0) + P(N_1\!=\!0)P(N_2\!=\!1)P(N_3\!=\!0) +$$
$$+P(N_1\!=\!0)P(N_2\!=\!0)P(N_3\!=\!1) = \frac{0.104}{3}.$$

Let V_j be the expected time for a customer who arrives to facility Fj to go through that facility once. Then $V_j = L_j/\lambda_j$, so that

$V_1 = (4/3)/20 = 1/15,\ V_2 = 4/24 = 1/6,\ V_3 = 4/24 = 1/6.$

Let W_j be the expected remaining time in the system for a customer who arrives to facility Fj. Then

$W_1 = V_1 + 0.4 \cdot W_2 + 0.3 \cdot W_3,$
$W_2 = V_2 + (3/4) \cdot W_3,$
$W_3 = V_3 + (2/3) \cdot W_2,$

which gives that

$W_1 = 7/15$ hours $= 28$ minutes,
$W_2 = 7/12$ hours $= 35$ minutes,
$W_3 = 5/9$ hours $= 33$ minutes and 20 seconds.

So the answer on (c) is $W_2 = 35$ minutes.

Exercise 3.

Let A be the event that the error is of type A,
let B be the event that the error is of type B,
let "A" be the event that the test says that the error is of type A,
let "B" be the event that the test says that the error is of type B.

Then the follwing probabilities are known:

$P(A) = 0.6, \quad P(B) = 0.4,$
$P(\text{“A”}|A) = 2/3, \quad P(\text{“B”}|A) = 1/3,$
$P(\text{“A”}|B) = 1/4, \quad P(\text{“B”}|B) = 3/4.$

But we will also need the probablities:

$P(\text{“A”}), \; P(\text{“B”}), \; P(A\,|\text{“A”}), \; P(B\,|\text{“A”}), \; P(A\,|\text{“B”}), \; P(B\,|\text{“B”}).$

These are obtained as follows:

$P(\text{“A”}) = P(\text{“A”} \cap A) + P(\text{“A”} \cap B) = P(A)\,P(\text{“A”}|A) + P(B)\,P(\text{“A”}|B) = 0.5.$
$P(\text{“B”}) = 1 - P(\text{“A”}) = 0.5.$

$$P(A\,|\text{“A”}) = \frac{P(A \cap \text{“A”})}{P(\text{“A”})} = \frac{P(A)\,P(\text{“A”}|A)}{P(\text{“A”})} = 0.8, \quad P(B\,|\text{“A”}) = 1 - P(A\,|\text{“A”}) = 0.2,$$

$$P(B\,|\text{“B”}) = \frac{P(B \cap \text{“B”})}{P(\text{“B”})} = \frac{P(B)\,P(\text{“B”}|B)}{P(\text{“B”})} = 0.6, \quad P(A\,|\text{“B”}) = 1 - P(B\,|\text{“B”}) = 0.4.$$

Now we are ready to analyze and compare the different decisions.

This is best illustrated by drawing a decision tree, but since we are reluctant to do this in latex, we present the solution in a slightly more boring way.

First, there is two alternatives, to test or not to test.

NOTEST
Assume that we skip the test. Then there are two alternatives,
to start with RA or to start with RB.

NOTEST - RA
Skip the test, do RA first and then, if needed, do RB.
Then the expected time to cure the engine is $P(A) \cdot 40 + P(B) \cdot 70 = 52$ hours.

NOTEST - RB
Skip the test, do RB first and then, if needed, do RA.
Then the expected time to cure the engine is $P(B) \cdot 30 + P(A) \cdot 70 = 54$ hours.

Thus, the minmal expected total time for the case **NOTEST** is 52 hours.

The solution continues on next page.

TEST
Assume that we start by doing the test. Then two things may happen, **"A"** or **"B"**.

TEST - "A"
Assume that the test says that the error is of type A.
The probability for this is $P(\text{"A"}) = 0.5$.
Then there are two possible ways to continue, to do RA (and if needed RB), or to do RB (and if needed RA).

TEST - "A" - RA
Assume that the test says that the error is of type A, and that we continue with RA.
Then the expected time to cure the engine is
$P(A\,|\,\text{"A"}) \cdot (40 + T) + P(B\,|\,\text{"A"}) \cdot (70 + T) = 46 + T$ hours.

TEST - "A" - RB
Assume that the test says that the error is of type A, and that we continue with RB.
Then the expected time to cure the engine is
$P(B\,|\,\text{"A"}) \cdot (30 + T) + P(A\,|\,\text{"A"}) \cdot (70 + T) = 62 + T$ hours.

Thus, the minmal expected total time for the case **TEST - "A"** is $46 + T$ hours.

TEST - "B"
Assume that the test says that the error is of type B.
The probability for this is $P(\text{"B"}) = 0.5$.
Then there are two possible ways to continue, to do RB (and if needed RA), or to do RA (and if needed RB).

TEST - "B" - RB
Assume that the test says that the error is of type B, and that we continue with RB.
Then the expected time to cure the engine is
$P(B\,|\,\text{"B"}) \cdot (30 + T) + P(A\,|\,\text{"B"}) \cdot (70 + T) = 46 + T$ hours.

TEST - "B" - RA
Assume that the test says that the error is of type B, and that we continue with RA.
Then the expected time to cure the engine is
$P(A\,|\,\text{"B"}) \cdot (40 + T) + P(B\,|\,\text{"B"}) \cdot (70 + T) = 58 + T$ hours.

Thus, the minmal expected total time for the case **TEST - "B"** is $46 + T$ hours.

The minmal expected total time for the case **TEST** is then given by
$P(\text{"A"}) \cdot (46 + T) + P(\text{"B"}) \cdot (46 + T) = 46 + T$ hours.

Finally, we see that the case **TEST** is better than the case **NOTEST** if $T < 6$, which means that we should do the test if it requires less than 6 hours to complete.

Exercise 4.
We have the following transistion probabilities:

$p_{SS}(A) = 3/4,\ \ p_{ST}(A) = 1/4,$
$p_{TS}(A) = 1/4,\ \ p_{TT}(A) = 3/4,$

$p_{SS}(B) = 1/2,\ \ p_{ST}(B) = 1/2,$
$p_{TS}(B) = 1/2,\ \ p_{TT}(B) = 1/2.$

The expected immediate cost for different actions in different states are then given by

$C_{SA} = p_{SS}(A)\,C(S{-}A{-}S) + p_{ST}(A)\,C(S{-}A{-}T) = (3/4)\cdot 24 + (1/4)\cdot 28 = 25.$
$C_{SB} = p_{SS}(B)\,C(S{-}B{-}S) + p_{ST}(B)\,C(S{-}B{-}T) = (1/2)\cdot 24 + (1/2)\cdot 40 = 32.$
$C_{TA} = p_{TS}(A)\,C(T{-}A{-}S) + p_{TT}(A)\,C(T{-}A{-}T) = (1/4)\cdot 14 + (3/4)\cdot 6 = 8.$
$C_{TB} = p_{TS}(B)\,C(T{-}B{-}S) + p_{TT}(B)\,C(T{-}B{-}T) = (1/2)\cdot 2 + (1/2)\cdot 0 = 1.$

(4.a):
First, the three numbers g, v_S and v_T, corresponding to the suggested policy $d_S = B$ and $d_T = A$, are calculated from the system

$v_T = 0,$
$g + v_S = C_{SB} + p_{SS}(B)\,v_S + p_{ST}(B)\,v_T,$
$g + v_T = C_{TA} + p_{TS}(A)\,v_S + p_{TT}(A)\,v_T,$

which becomes

$v_T = 0,$
$g + v_S = 32 + 0.5\,v_S + 0.5\,v_T,$
$g + v_T = 8 + 0.25\,v_S + 0.75\,v_T,$

with the unique solution $g = 16$, $v_S = 32$, $v_T = 0$.

To check optimality of the current policy, we first check if

$g + v_S = \min\{\ C_{SA} + p_{SS}(A)\,v_S + p_{ST}(A)\,v_T,\ C_{SB} + p_{SS}(B)\,v_S + p_{ST}(B)\,v_T\ \}.$

The left hand side is $16 + 32 = 48$, while the right hand side is
$\min\{\ 25 + 0.75\cdot 32 + 0.25\cdot 0\,,\ 32 + 0.5\cdot 32 + 0.5\cdot 0\ \} = \min\{\ 49,\ 48\ \} = 48$, OK!

Next, we check if

$g + v_T = \min\{\ C_{TA} + p_{TS}(A)\,v_S + p_{TT}(A)\,v_T,\ C_{TB} + p_{TS}(B)\,v_S + p_{TT}(B)\,v_T\ \}.$

The left hand side is $16 + 0 = 16$, while the right hand side is
$\min\{\ 8 + 0.25\cdot 32 + 0.75\cdot 0\,,\ 1 + 0.5\cdot 32 + 0.5\cdot 0\ \} = \min\{\ 16,\ 17\ \} = 16$, OK!

Thus, the suggested policy $d_S = B$ and $d_T = A$ is optimal.

4.(b):
The LP problem without discounting is

$$\begin{aligned}
\text{minimize} \quad & 25\,Y_{SA} + 32\,Y_{SB} + 8\,Y_{TA} + Y_{TB} \\
\text{subject to} \quad & Y_{SA} + Y_{SB} + Y_{TA} + Y_{TB} = 1, \\
& Y_{SA} + Y_{SB} - p_{SS}(A)\,Y_{SA} - p_{SS}(B)\,Y_{SB} - p_{TS}(A)\,Y_{TA} - p_{TS}(B)\,Y_{TB} = 0, \\
& Y_{TA} + Y_{TB} - p_{ST}(A)\,Y_{SA} - p_{ST}(B)\,Y_{SB} - p_{TT}(A)\,Y_{TA} - p_{TT}(B)\,Y_{TB} = 0, \\
& Y_{SA},\ Y_{SB},\ Y_{TA},\ Y_{TB} \geq 0.
\end{aligned}$$

which becomes

$$\begin{aligned}
\text{minimize} \quad & 25\,Y_{SA} + 32\,Y_{SB} + 8\,Y_{TA} + Y_{TB} \\
\text{subject to} \quad & Y_{SA} + Y_{SB} + Y_{TA} + Y_{TB} = 1, \\
& 0.25\,Y_{SA} + 0.5\,Y_{SB} - 0.25\,Y_{TA} - 0.5\,Y_{TB} = 0, \\
& (-0.25\,Y_{SA} - 0.5\,Y_{SB} + 0.25\,Y_{TA} + 0.5\,Y_{TB} = 0,) \\
& Y_{SA},\ Y_{SB},\ Y_{TA},\ Y_{TB} \geq 0.
\end{aligned}$$

The last two equations are the same so one may be removed, as indicated by the parenthesis. The results from (a) imply that the only non-zero variables in the optimal solution of this LP problem are Y_{SB} and Y_{TA}.
The first two equations then give that $Y_{SB} + Y_{TA} = 1$ and $0.5\,Y_{SB} - 0.25\,Y_{TA} = 0$, so that $Y_{SB} = 1/3$ and $Y_{TA} = 2/3$, while $Y_{SA} = Y_{TB} = 0$.
The optimal value of the LP problem is $32 \cdot (1/3) + 8 \cdot (2/3) = 16$.

(4.c):
First, the two numbers V_S and V_T, corresponding to the suggested policy $d_S = B$ and $d_T = A$, are calculated from the system
$V_S = C_{SB} + 0.8 \cdot (p_{SS}(B)\,V_S + p_{ST}(B)\,V_T)$,
$V_T = C_{TA} + 0.8 \cdot (p_{TS}(A)\,V_S + p_{TT}(A)\,V_T)$,
which becomes
$V_S = 32 + 0.4\,V_S + 0.4\,V_T$,
$V_T = 8 + 0.2\,V_S + 0.6\,V_T$,
with the unique solution $V_S = 100$, $V_T = 70$.

To check optimality of the current policy, we first check if
$V_S = \min\{\, C_{SA} + 0.8 \cdot (p_{SS}(A)\,V_S + p_{ST}(A)\,V_T),\ C_{SB} + 0.8 \cdot (p_{SS}(B)\,V_S + p_{ST}(B)\,V_T) \,\}$.
The left hand side is 100, while the right hand side is
$\min\{\, 25 + 0.6 \cdot 100 + 0.2 \cdot 70\,,\ 32 + 0.4 \cdot 100 + 0.4 \cdot 70 \,\} = \min\{\, 99,\ 100 \,\} = 99$,
so the suggested policy is not optimal!

We could stop here, but just for fun we could also check if
$V_T = \min\{\, C_{TA} + 0.8 \cdot (p_{TS}(A)\,V_S + p_{TT}(A)\,V_T),\ C_{TB} + 0.8 \cdot (p_{TS}(B)\,V_S + p_{TT}(B)\,V_T) \,\}$.
The left hand side is 70, while the right hand side is
$\min\{\, 8 + 0.2 \cdot 100 + 0.6 \cdot 70\,,\ 1 + 0.4 \cdot 100 + 0.4 \cdot 70 \,\} = \min\{\, 70,\ 69 \,\} = 69$,
which once more shows that the suggested policy $d_S = B$ and $d_T = A$ is not optimal.
A strictly better policy is, according to the above calculations, $d_S = A$ and $d_T = B$.

Exercise 5.

Assume that the false coin is known to be among n specific coins.
If Hook puts k coins in each bowl, where $k \geq 1$ and $2k \leq n$, then one of the following two things will happen.
1. The two bowls contain equal weights, which happens with probability $(n-2k)/n$, in which case the false coin is among the left out $n-2k$ coins.
2. The two bowls contain different weights, which happens with probability $2k/n$, in which case the false coin is among the k coins in the lightest bowl.

This leads to the recursive equation:

$$V(n) = 1 + \min_k \{ \frac{n-2k}{n} \cdot V(n-2k) + \frac{2k}{n} \cdot V(k) \}, \text{ where } k \text{ must satisfy } 1 \leq k \leq \frac{n}{2}.$$

Obvious boundary condition is $V(1) = 0$.
Further, we may define $V(0)$ arbitrary, e.g. $V(0) = 0$,
since $V(0)$ is always multiplied by the probabililty 0.

This gives:

$V(2) = 1 + 0 \cdot V(0) + 1 \cdot V(1) = 1.$ Optimal $k = 1$.

$V(3) = 1 + \frac{1}{3} \cdot V(1) + \frac{2}{3} \cdot V(1) = 1.$ Optimal $k = 1$.

$V(4) = 1 + \min \{ \frac{2}{4} \cdot V(2) + \frac{2}{4} \cdot V(1),\ \frac{0}{4} \cdot V(0) + \frac{4}{4} \cdot V(2) \} = 1 + \min \{ \frac{1}{2},\ 1 \} = \dfrac{3}{2}$
Optimal $k = 1$.

$V(5) = 1 + \min \{ \frac{3}{5} \cdot V(3) + \frac{2}{5} \cdot V(1),\ \frac{1}{5} \cdot V(1) + \frac{4}{5} \cdot V(2) \} = 1 + \min \{ \frac{3}{5},\ \frac{4}{5} \} = \dfrac{8}{5}$
Optimal $k = 1$.

$V(6) = 1 + \min \{ \frac{4}{6} \cdot V(4) + \frac{2}{6} \cdot V(1),\ \frac{2}{6} \cdot V(2) + \frac{4}{6} \cdot V(2),\ \frac{0}{6} \cdot V(0) + \frac{6}{6} \cdot V(3) \} = 2$
Optimal $k = 1$, 2 or 3.

$V(7) = 1 + \min \{ \frac{5}{7} \cdot V(5) + \frac{2}{7} \cdot V(1),\ \frac{3}{7} \cdot V(3) + \frac{4}{7} \cdot V(2),\ \frac{1}{7} \cdot V(1) + \frac{6}{7} \cdot V(3) \} = \dfrac{13}{7}$
Optimal $k = 3$.

So the optimal strategy for Captain Hook is to first put 3 coins in each bowl. If the weights are equal, the left out coin is the false one. Otherwise, there are 3 coins to choose between, in which case one more balancing is needed, with one coin in each bowl and one left out.

By the way, note that $V(7) < V(6)$!