CHAPTER 3

The disturbance decoupling problem (DDP)

3.1. Geometric formulation

Consider the system

$$\begin{cases} \dot{x} &= Ax + Bu + Ew \\ y &= Cx. \end{cases}$$

Problem 3.1 (Disturbance decoupling). *Find a state feedback*

$$u = Fx + v$$

such that the output y is unaffected by the disturbance w, namely for any w_1,

$$y|_{w=w_1} = y|_{w=0}.$$

Under the feedback

$$u = Fx + v$$

the closed loop system becomes

$$\begin{cases} \dot{x} &= (A + BF)x + Bv + Ew \\ y &= Cx. \end{cases} \tag{3.1}$$

Assuming $x(0) = 0$, the equations (3.1) can be solved as

$$y(t) = \int_0^t Ce^{(A+BF)(t-s)}Bv(s)\,ds + \int_0^t Ce^{(A+BF)(t-s)}Ew(s)\,ds. \tag{3.2}$$

The requirement for disturbance decoupling is now that the last term in (3.2) be zero for any w. Or equivalently, in any derivative $y^{(i)}$ $i = 1, 2, \cdots$, w should not appear. For the sake of simplicity, let us first assume $v = 0$. Since

$$y^{(1)}(t) = C(A + BF)x(t) + CEw(t),$$

we must have $CE = 0$. Deductively, under the assumption $C(A+BF)^{i-2}E = 0,\ i \geq 2$, we have

$$y^{(i)} = C(A + BF)^i x + C(A + BF)^{i-1}Ew.$$

Thus we must have

$$\left\{\begin{array}{rcl} CE & = & 0 \\ C(A+BF)E & = & 0 \\ C(A+BF)^2E & = & 0 \\ & \vdots & \\ C(A+BF)^{n-1}E & = & 0 \end{array}\right. \tag{3.3}$$

The reason that we only need to have the first n identities is the Cayley-Hamilton theorem. The system is nonlinear in F and looks quite complicated! However, (3.3) can be stated as

$$\left\{\begin{array}{rcl} \operatorname{Im} E & \subseteq & \ker C \\ (A+BF)\operatorname{Im} E & \subseteq & \ker C \\ (A+BF)^2\operatorname{Im} E & \subseteq & \ker C \\ & \vdots & \\ (A+BF)^{n-1}\operatorname{Im} E & \subseteq & \ker C, \end{array}\right. \tag{3.4}$$

which is equivalent to

$$\operatorname{Im} E + (A+BF)\operatorname{Im} E + \ldots + (A+BF)^{n-1}\operatorname{Im} E \subseteq \ker C,$$

i.e.,

$$\langle A+BF|\operatorname{Im} E\rangle \subseteq \ker C. \tag{3.5}$$

The subspace $\mathcal{V} := \langle A+BF|\operatorname{Im} E\rangle$ has the invariance property

$$(A+BF)\mathcal{V} \subseteq \mathcal{V}, \tag{3.6}$$

i.e., it is $(A+BF)$-invariant, or controlled invariant.

Hence, the disturbance decoupling problem has now been translated into the following geometric problem with linear structure.

PROBLEM 3.2. *Find a subspace $\mathcal{V}$ such that*

(1) $\operatorname{Im} E \subseteq \mathcal{V} \subseteq \ker C$
(2) $\mathcal{V}$ *is controlled invariant.*

We now proceed with the solution of the disturbance decoupling problem, as formulated in Problem 3.2. It turns out that among all (A, B)-invariant subspaces contained in $\ker C$ (or any a priori given subspace $\mathcal{Z}$) there is a *maximal* one in the sense of set inclusion, as we discussed in the previous chapter.

The solvability of the disturbance decoupling problem can now be formulated in terms of $\mathcal{V}^* := \mathcal{S}^*(\ker C)$.

THEOREM 3.1. *The disturbance decoupling problem is solvable if and only if*

$$\operatorname{Im} E \subseteq \mathcal{V}^*.$$

PROOF
(if) Since $\mathcal{V}^*$ is (A, B)-invariant, by Theorem 2.2, there is an F such that $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$. If Im $E \subseteq \mathcal{V}^*$, then

$$\langle A + BF | \operatorname{Im} E\rangle \subseteq \langle A + BF | \mathcal{V}^*\rangle = \mathcal{V}^* \subseteq \ker C.$$

(only if) For disturbance decoupling it is necessary that

$$\mathcal{V} := \langle A + BF | \operatorname{Im} E\rangle \subseteq \ker C,$$

as we have seen above. Clearly, $\mathcal{V}$ is an (A, B)-invariant subspace of $\ker C$, and therefore $\mathcal{V} \subseteq \mathcal{V}^*$. Then, Im $E \subseteq \mathcal{V}$ implies Im $E \subseteq \mathcal{V}^*$. ∎

COROLLARY 3.2. *A necessary condition for solvability of the disturbance decoupling problem is $CE = 0$.*

What are the consequences of this corollary?

3.2. Computing $\mathcal{V}^*$

Next we turn to the problem of computing $\mathcal{S}^*(\ker C)$ (or in general, of any subspace $\mathcal{Z}$).

THEOREM 3.3 ($\mathcal{V}^*$-algorithm). *Let $\mathcal{V}_0 := \ker C$ and define, for $i = 0, 1, 2, \ldots$*

$$\mathcal{V}_{i+1} = \{x \in \ker C \mid Ax \in \mathcal{V}_i + \operatorname{Im} B\}.$$

Then $\mathcal{V}_{i+1} \subset \mathcal{V}_i$, and for some integer $q \leq \dim \mathcal{V}_0$,

$$\mathcal{V}_i = \mathcal{V}^* \quad \textit{for all } i \geq q.$$

PROOF
We show that $\mathcal{V}_{i+1} \subset \mathcal{V}_i$ by induction. It is obvious that $\mathcal{V}_1 \subset \mathcal{V}_0$. Suppose that $\mathcal{V}_i \subset \mathcal{V}_{i-1}$. Then

$$\begin{aligned}\mathcal{V}_{i+1} &= \{x \in \ker C \mid Ax \in \mathcal{V}_i + \operatorname{Im} B\} \\ &\subset \{x \in \ker C \mid Ax \in \mathcal{V}_{i-1} + \operatorname{Im} B\} = \mathcal{V}_i.\end{aligned}$$

Since $\mathcal{V}_{i+1} \neq \mathcal{V}_i$ if and only if $\dim \mathcal{V}_{i+1} < \dim \mathcal{V}_i$, the algorithm must converge in q steps for some $q \leq \dim \mathcal{V}_0$, i.e., $\mathcal{V}_i = \mathcal{V}_q$ for $i \geq q$. Therefore,

$$\mathcal{V}_q = \{x \in \ker C \mid Ax \in \mathcal{V}_q + \operatorname{Im} B\}$$

i.e., $A\mathcal{V}_q \subset \mathcal{V}_q + \operatorname{Im} B$ and $\mathcal{V}_q \subset \ker C$. Thus, $\mathcal{V}_q$ is an (A, B)-invariant subspace in $\ker C$, i.e., $\mathcal{V}_q \in S(\ker C)$. Let $\mathcal{V} \in S(\ker C)$ be arbitrary. Then, $\mathcal{V} \subset \mathcal{V}_0$, and if $\mathcal{V} \subset \mathcal{V}_i$,

$$\begin{aligned}\mathcal{V} &\subset \{x \in \ker C \mid Ax \in \mathcal{V} + \operatorname{Im} B\} \\ &\subset \{x \in \ker C \mid Ax \in \mathcal{V}_i + \operatorname{Im} B\} = \mathcal{V}_{i+1}\end{aligned}$$

which implies that $\mathcal{V} \subset \mathcal{V}_q$. Therefore, since $\mathcal{V}$ is arbitrary, $\mathcal{V}_q = \mathcal{V}^*$.

∎

However, the above procedure is not very convenient for computing $\mathcal{V}^*$. It turns out that the kernel to $\mathcal{V}^*$ is much easier to compute. In the literature, it is sometimes called the Ω^* algorithm. We will introduce this algorithm shortly.

The above conceptual algorithm can be concretized into matrix computations form. Here we show how to transform the algorithm into matrix computations, and in Appendix A we indicate some further numerical aspects to be aware of. Let $\{v_1, v_2, \ldots, v_{q_i}\}$ be a basis in $\mathcal{V}_i$ and define the column stacked matrix

$$V_i = [v_1 \, v_2 \, \ldots \, v_{q_i}]$$

Then,

$$\mathcal{V}_i + \mathrm{Im}\ B = \mathrm{Im}\,[V_i \quad B] = (\ker[V_i \quad B]')^\perp$$

and therefore the recursion of Theorem 3.3 can be written

$$\begin{aligned}\mathcal{V}_{i+1} &= \ker C \cap \{x \mid Ax \perp \ker[V_i \quad B]'\} \\ &= \ker C \cap \{x \mid z_j' Ax = 0, j = 1, 2, \ldots, p_i\}\end{aligned}$$

if $\{z_1, z_2, \ldots, z_{p_i}\}$ is a basis of $\ker[V_i \quad B]'$, i.e., a maximal number of linearly independent vectors such that

$$[V_i \quad B]' z_j = 0 \quad \text{or} \quad z_j'[V_i \quad B] = 0$$

If we set

$$Z_i = \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_{p_i}' \end{bmatrix}$$

the recursion in Theorem 3.3 can be written as

$$\mathcal{V}_{i+1} = \ker C \cap \ker[Z_i A]$$

where Z_i is a matrix solution of

$$Z_i[V_i \quad B] = 0$$

with a maximal number of rows, i.e, the rows of Z_i are a basis for the left null space of $[V_i \quad B]$. Thus, the columns of every maximal matrix solution V_{i+1} of

$$\begin{bmatrix} C \\ Z_i A \end{bmatrix} V_{i+1} = 0$$

(i.e. with a maximal number of columns) form a basis in $\mathcal{V}_{i+1}$.

From the above derivation, we can summarize the Ω^* algorithm as follows.

Ω^*-*algorithm:*

Denote $G = \mathrm{Im}\ B$.

- Step 0: $\Omega_0 = Span\{C\}$,
- Step k: $\Omega_k = \Omega_{k-1} + L_{Ax}(\Omega_{k-1} \cap G^\perp)$. Where $L_{Ax}(\Omega \cap G^\perp)$ is the span of all row vectors ωA where $\omega \in \Omega \cap G^\perp$.

- If there is a k^* such that $\Omega_{k^*+1} = \Omega_{k^*}$, then
$$\mathcal{V}^* = \Omega_{k^*}^{\perp}.$$

EXAMPLE 3.1. *Let* $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *and* $C = [1\ 0]$.
Then $G^{\perp} = span\{[1\ 0]\}$, *and*
$$\Omega_1 = spanC + span\{[1\ 0]A\} = span\{[1\ 0], [0\ 1]\}.$$
Since Ω_1 *has full rank, thus* $\mathcal{V}^* = 0$.

i.e., no disturbance decoupling can occur even if $CE = 0$.

Take, now A *and* B *as above and* $C = [1, -1]$. *Then,*
$$\Omega_1 = spanC + span\{[0\ 0]A\} = span\{[1\ -1]\} = \Omega_0.$$
Therefore,
$$\mathcal{V}^* = \Omega_1^{\perp} = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

The feedback which achieves disturbance decoupling was computed previously in Example 2.2.

3.3. Disturbance decoupling, observability, and zeros

We shall now show that the idea behind disturbance decoupling by state feedback is to introduce unobservability in the closed-loop system. We should point out that this can only be done with full state feedback. No observer can be used here since we do not assume any information on the disturbances.

Recall that for a pair (C, A) the unobservable subspace is defined as $\ker \Omega$, where Ω is the observability matrix of (C, A). The unobservable subspace is A-invariant, in fact, it is the *maximal* A-invariant subspace in $\ker C$.

Suppose that (A, B, C) is minimal. To achieve disturbance decoupling by means of the control law $u = Fx + v$ we seek an F that produces the maximal $A+BF$-invariant subspace in $\ker C$, i.e., we maximize with respect to F the unobservable subspace of the pair $(C, A + BF)$. As a consequence, if $F \in \mathcal{F}(\mathcal{V}^*)$ and $\mathcal{V}^*$ is nontrivial then the realization $(A + BF, B, C)$ is no longer minimal. Note, however, that the pair $(A + BF, B)$ is still reachable.

Consider the special case of a SISO-system. If $(A + BF, B, C)$ is *not* a minimal realization of the scalar transfer function
$$C(sI - A - BF)^{-1}B$$
then pole/zero cancellation takes place. Conversely, since the denominator polynomial can be arbitrarily assigned by state feedback, any zero of the original transfer function can be canceled. Thus, it is the presence of zeros in the transfer function $W(s) =: C(sI - A)^{-1}B$ that makes $\mathcal{V}^*$ nontrivial. Hence, for a SISO-system it follows that
$$\dim \mathcal{V}^* = \#\{\text{zeros of } W(s)\}.$$

Moreover, from this we conclude something about disturbance decoupling *with stability.* If $W(s)$ has a zero in the right half plane, then disturbance decoupling requires that we place a pole in the right half plane, and $A+BF$ will not be a stability matrix. This is an example of the general fact that presence of zeros in the right half plane makes a system difficult to control.

In the multivariable case, treated in Chapter 4, the situation is a little bit more delicate, but the basic idea remains the same.

We illustrate the preceding discussion with an example.

EXAMPLE 3.2. *Let* $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *and* $C = [1, -1]$.
In Example 2.2 we showed that $\ker C = \{x \in \mathbb{R}^2 : x_1 = x_2\}$ *is* (A, B)*-invariant. Hence,* $\mathcal{V}^* = \{x \in \mathbb{R}^2 : x_1 = x_2\}$. *Moreover, we know that* $F = \begin{bmatrix} -1 & -1 \end{bmatrix} \in \mathcal{F}(\mathcal{V}^*)$.

The original transfer function is

$$C(sI - A)^{-1}B = \frac{s-1}{s^2 - s - 2},$$

and

$$A + BF = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C(sI - A - BF)^{-1}B = -\frac{s-1}{s^2-1} = -\frac{1}{s+1}.$$

Moreover,

$$\begin{bmatrix} C \\ C(A+BF) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

so $(C, A+BF)$ *is not observable.*

3.4. Disturbance decoupling with eigenvalue assignment

Disturbance decoupling alone is a property of limited value. Of utmost concern in the design of any control law is to guarantee that the closed loop system behaves in a reasonable way. The first requirement is clearly to insure stability of the system, but often this is not enough. A desirable asset is to place the poles of the closed loop system at arbitrary positions; the system can then have better damping, rise time and many other properties. This objective is formulated in the *disturbance decoupling with pole placement* (DDPP) or *eigenvalue assignment.* Consider the system

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu + Ew \\ y & = & Cx \end{array} \right. \tag{3.7}$$

where (A, B) is reachable. Suppose now that we want to find a feedback law $u = Fx + Gv$ such that

(i) y is unaffected by w
(ii) the closed-loop system is stable.

Since the closed-loop system is

$$\dot{x} = (A + BF)x + BGv + Ew$$

the solution depends only on the choice of F and not on G. We showed earlier that condition (i) is equivalent to finding a subspace $\mathcal{V} \in S(\ker C)$ such that $\mathrm{Im}\ E \subseteq \mathcal{V}$. Any friend $F \in \mathcal{F}(\mathcal{V})$ then solves problem (i). For example, we can choose $\mathcal{V} = \mathcal{V}^*$. If we are only interested in satisfying condition (i), we may choose $\mathcal{V} = \mathcal{V}^*$, a choice which gives the weakest possible conditions but also fewer friends. However, a feedback law provided by a friend of $\mathcal{V}^*$ may not stabilize the system, i.e., satisfy condition (ii). We shall now study this question closer.

If we want to satisfy (i) and (ii) simultaneously we must in general choose F from a larger class of feedback laws, i.e., choose a smaller $\mathcal{V}$. It turns out that $\mathcal{V} = \mathcal{R}^*$ works. We have

$$\mathcal{R}^* \subseteq \mathcal{V}^*$$

and from Theorem 2.8 it follows that $\mathcal{R}^*$ has more friends than $\mathcal{V}^*$,

$$\mathcal{F}(\mathcal{R}^*) \supseteq \mathcal{F}(\mathcal{V}^*).$$

3.5. Solution of the DDPP

Theorem 3.4. *Consider the system* (3.7) *and let* $r = \dim \mathcal{R}^*$. *Let* ϕ *and* ψ *be polynomials with real coefficients such that*

$$\phi(s) = s^r + \phi_1 s^{r-1} + \ldots + \phi_r$$

and

$$\psi(s) = s^{n-r} + \psi_1 s^{n-r-1} + \ldots + \psi_{n-r}.$$

Then, if

$$\mathrm{Im}\ E \subseteq \mathcal{R}^* \tag{3.8}$$

there is an $F \in \mathcal{F}(\mathcal{R}^*)$ *satisfying condition (i) such that* $(A + BF)$ *has characteristic polynomial*

$$\chi_{A+BF} = \phi\psi. \tag{3.9}$$

In particular, we can satisfy condition (ii) by letting ϕ *and* ψ *be stable.*

Proof

As pointed out above, disturbance decoupling will be achieved for any $F \in \mathcal{F}(\mathcal{R}^*)$ provided that (3.8) is fulfilled. Therefore it only remains to prove that there is an $F \in \mathcal{F}(\mathcal{R}^*)$ such that (3.9) holds. To this end, express the state space as a direct sum

$$\mathbb{R}^n = \mathcal{R}^* \oplus \mathcal{W}, \tag{3.10}$$

where $\mathcal{W}$ is chosen so that

$$\mathrm{Im}\ B = (\mathrm{Im}\ B) \cap \mathcal{R}^* \oplus (\mathrm{Im}\ B) \cap \mathcal{W}. \tag{3.11}$$

(Warning: such a decomposition is not valid for arbitrary choice of $\mathcal{W}$. Why?) Let $\{p_1, \dots, p_q\}$ be a basis for $\operatorname{Im} B \cap \mathcal{R}^*$ and choose an invertible

$$G = \begin{array}{ccc} [& G_1 & G_2 &] \\ & q & k-q & \end{array}$$

such that $\operatorname{Im} BG_1 = \operatorname{Im} B \cap \mathcal{R}^*$ and $\operatorname{Im} BG_2 \subset \mathcal{W}$. From now on we work with the new matrix BG replacing the B, but we still call it B. Let the basis $\{p_1, p_2, \dots, p_n\}$ for $\mathbb{R}^n$ be adapted to the decomposition (3.10) in the sense that $\{p_1, \dots, p_q, \dots, p_r\}$ is a basis for $\mathcal{R}^*$.

In this basis the system has the following structure

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $x_1(t) \in \mathcal{R}^*$ (of dimension r) and $x_2(t) \in \mathcal{W}$ (of dimension $n-r$).

Let $F \in \mathcal{F}(\mathcal{R}^*)$ and take $u = Fx + v$. Partitioning F as

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \text{ yields } BF = \begin{bmatrix} B_1F_{11} & B_1F_{12} \\ B_2F_{21} & B_2F_{22} \end{bmatrix}.$$

Since $(A + BF)\mathcal{R}^* \subseteq \mathcal{R}^*$, it follows that

$$A + BF = \begin{bmatrix} A_{11} + B_1F_{11} & A_{12} + B_1F_{12} \\ 0 & A_{22} + B_2F_{22} \end{bmatrix},$$

from where

$$A_{21} + B_2F_{21} = 0. \tag{3.12}$$

F is a friend of $\mathcal{R}^*$ for any choice of F_{11}, F_{12} and F_{22} as long as F_{21} satifies (3.12).

The characteristic polynomial of $A + BF$ is

$$\det(sI - A - BF) = \tag{3.13}$$

$$\det(sI - A_{11} - B_1F_{11}) \det(sI - A_{22} - B_2F_{22}).$$

Next we show that F_{11}, F_{22} can be chosen so that (3.13) becomes $\phi(s)\psi(s)$.

The closed-loop system has the structure

$$\begin{cases} \dot{x}_1 &= (A_{11} + B_1F_{11})x_1 + (A_{21} + B_1F_{12})x_2 + B_1v_1 \\ \dot{x}_2 &= (A_{22} + B_2F_{22})x_2 + B_2v_2. \end{cases} \tag{3.14}$$

Since $\mathcal{R}^*$ is a reachability subspace, (A_{11}, B_1) is reachable, and we can choose F_{11} so that

$$\det(sI - A_{11} - B_1F_{11}) = \phi(s).$$

Moreover, since (A, B) is reachable, then so is (A_{22}, B_2) (why?), and therefore we can choose F_{22} so that

$$\det(sI - A_{22} - B_2F_{22}) = \psi(s).$$

Consequently, (3.9) is fulfilled as required.

EXAMPLE 3.3. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Can we achieve disturbance decoupling with stability if

$$\text{(a) } E = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}; \quad \text{(b) } E = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}; \quad \text{(c) } E = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

In Example 2.5 we showed that

$$\mathcal{V}^* = \text{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathcal{R}^* = \text{Im} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

Therefore, the answers are:

(a) *Yes, since* Im $E \subseteq \mathcal{R}^*$.
(b) *We can achieve disturbance decoupling, since* Im $E \subset \mathcal{V}^*$, *but,* perhaps *not stability since* Im $E \not\subseteq \mathcal{R}^*$.
(c) *No, not even disturbance decoupling, since* Im $E \not\subseteq \mathcal{V}^*$.

3.6. Is it necessary that Im $E \subset \mathcal{R}^*$?

The somewhat unclear answer in case (b) above motivates further study of the following question. When can we achieve stability with an

$$F \in \mathcal{F}(\mathcal{V}^*) \subseteq \mathcal{F}(\mathcal{R}^*)?$$

Let $\mathcal{V}^*/\mathcal{R}^*$ be an arbitrary subspace such that

$$\mathcal{V}^* = \mathcal{R}^* \oplus \mathcal{V}^*/\mathcal{R}^*.$$

Abstractly, $\mathcal{V}^*/\mathcal{R}^*$ is a *quotient space*, hence the somewhat cumbersome notation. Let $\{p_1, \ldots, p_n\}$ be a basis in $\mathbb{R}^n$ that is adapted to the decomposition

$$\mathbb{R}^n = \mathcal{R}^* \oplus \mathcal{V}^*/\mathcal{R}^* \oplus \mathcal{W}$$

in the sense that $\{p_1, \ldots, p_r\}$ is a basis in $\mathcal{R}^*$, $\{p_{r+1}, \ldots, p_\nu\}$ in $\mathcal{V}^*/\mathcal{R}^*$ and $\{p_{\nu+1}, \ldots, p_n\}$ in $\mathcal{W}$, and for simplicity, the first q columns of B are a basis for Im $B \cap \mathcal{R}^*$. This decomposition is analogous to the one in (3.10) and (3.11).

Then the system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$$

can be rewritten in this basis as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y &= \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix} x, \end{aligned} \tag{3.15}$$

where $x_1(t) \in \mathcal{R}^*$, $x_2(t) \in \mathcal{V}^*/\mathcal{R}^*$ and $x_3(t) \in \mathcal{W}$ and the columns of B are partitioned according to (3.11). The zero blocks in B and C are consequences of

$$\mathrm{Im} \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} = \mathrm{Im}\ B \cap \mathcal{V}^* \subseteq \mathcal{R}^*, \ \mathrm{Im} \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix} = \mathrm{Im}\ B \cap \mathcal{W} \subseteq \mathcal{W},$$

and $\mathcal{V}^* \subseteq \ker C$.

Partitioning $F \in \mathcal{F}(\mathcal{V}^*)$ as

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \end{bmatrix}$$

we have

$$BF = \begin{bmatrix} B_1F_{11} & B_1F_{12} & B_1F_{13} \\ 0 & 0 & 0 \\ B_3F_{21} & B_3F_{22} & B_3F_{23} \end{bmatrix}.$$

Now, since we also have $F \in \mathcal{F}(\mathcal{R}^*)$, $(A+BF)\mathcal{R}^* \subseteq \mathcal{R}^*$, and therefore

$$(A+BF) \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (A_{11}+B_1F_{11})x_1 \\ A_{21}x_1 \\ (A_{31}+B_3F_{21})x_1 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

for all x_1. Consequently, $A_{21} = 0$ and $A_{31} + B_3F_{21} = 0$, fixing F_{21}.

Likewise, the invariance $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ implies that

$$(A+BF) \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (A_{12}+B_1F_{12})x_2 \\ A_{22}x_2 \\ (A_{32}+B_3F_{22})x_2 \end{bmatrix} = \begin{bmatrix} * \\ ** \\ 0 \end{bmatrix}$$

for all x_2 so that $A_{32} + B_3F_{22} = 0$, determining F_{22}.

Thus,

$$A+BF = \begin{bmatrix} A_{11}+B_1F_{11} & A_{12}+B_1F_{12} & A_{13}+B_1F_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33}+B_3F_{23} \end{bmatrix}$$

with the characteristic polynomial

$$\phi_1(s)\,\phi_2(s)\,\phi_3(s) = \det(sI - A_{11} - B_1F_{11})\ \det(sI - A_{22})\ \det(sI - A_{33} - B_3F_{23}).$$

Since (A, B) is reachable, (A_{33}, B_3) is reachable, and consequently $\phi_3(s)$ can be chosen arbitrarily by a suitable choice of F_{23}. Moreover, since $\mathcal{R}^*$ is a reachability subspace, $\phi_1(s)$ can also be chosen arbitrarily by a suitable choice of F_{11}. However, we cannot affect the eigenvalues of A_{22}.

Hence, we can stabilize the system with an $F \in \mathcal{F}(\mathcal{V}^*)$ only if A_{22} is stable.

It turns out that the eigenvalues A_{22} are the *zeros* of the multivariable transfer function $W(s) = C(sI - A)^{-1}B$, in a sense to be defined in the next chapter.