Mathematical Systems Theory: Advanced Course
Exercise Session 5

1 Accessibility of a nonlinear system

Consider an affine nonlinear control system:

$$\dot{x} = f(x) + G(x)u,\ x(0) = x_0, \quad G(x) = \left[\begin{array}{ccc} g_1(x) & \cdots & g_m(x) \end{array}\right],$$

where $x \in N \subset R^n$, N is an open set and $u \in R^m$. We will discuss the accessibility of this system, which is a weaker concept than the controllability.

Definition

The system is called *locally strongly accessible from* x_0 if for any initial point in the neighborhood of x_0, the set of reachable points with appropriate u contains a non-empty open set for any sufficiently small final time T.

Proposition

If $\dim \mathcal{R}_c(x_0) = n$, then the system is locally strongly accessible from x_0, where $R_c(x)$ is the *strong accessibility distribution* (see page 66 in the lecture note).

Procedure to compute $\mathcal{R}_c(x)$

Step 1. Take

$$\mathcal{R}_0(x) = \operatorname{span}\{g_1(x), \cdots, g_m(x)\}.$$

Set $k = 0$.

Step 2. Compute Lie brackets

$$[f, d],\ [g_i, d],\ \forall d(x) \in \mathcal{R}_k(x),$$

and take

$$\mathcal{R}_{k+1}(x) = \mathcal{R}_k(x) + \operatorname{span}\{\text{Lie brackets which are not in } \mathcal{R}_k(x)\}.$$

Step 3. Stop and set $\mathcal{R}_c(x) = \mathcal{R}_{k+1}(x)$ if $\mathcal{R}_{k+1}(x) = \mathcal{R}_k(x)$, or $\dim \mathcal{R}_{k+1}(x) = n, \forall x \in N$. Otherwise, return to Step 2 with $k = k + 1$.

Note: There is no guarantee that the process will end up.

Example

Consider the angular motion of a spacecraft. Here we assume there are only two controls (two pairs of boosters) available. The model for angular velocities around the three principal axes is as follows:

$$\begin{aligned}
\dot{x}_1 &= \frac{a_2 - a_3}{a_1} x_2 x_3 \\
\dot{x}_2 &= \frac{a_3 - a_1}{a_2} x_1 x_3 + u_1 \\
\dot{x}_3 &= \frac{a_1 - a_2}{a_3} x_2 x_1 + u_2 \\
& a_1 > 0,\ a_2 > 0,\ a_3 > 0.
\end{aligned}$$

Let us compute the strong accessibility distribution $R_c(x)$ and check the accessibility of the system. In this case,

$$f(x) := \begin{bmatrix} \alpha x_2 x_3 \\ \beta x_3 x_1 \\ \gamma x_1 x_2 \end{bmatrix},\ g_1(x) := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ g_2(x) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

where $\alpha := (a_2 - a_3)/a_1$, $\beta = (a_3 - a_1)/a_2$ and $\gamma = (a_1 - a_2)/a_3$.

Step 1. $\mathcal{R}_0(x) = \operatorname{span}\{g_1(x), g_2(x)\} = \operatorname{span}\{e_2, e_3\}$.

Step 2. Lie brackets are computed as follows:

$$\begin{aligned}
[f, g_1] &= \frac{\partial e_2}{\partial x} f(x) - \frac{\partial f}{\partial x} e_2 = - \begin{bmatrix} \alpha x_3 \\ 0 \\ \gamma x_1 \end{bmatrix} =: g_3(x) \\
[f, g_2] &= \frac{\partial e_3}{\partial x} f(x) - \frac{\partial f}{\partial x} e_3 = - \begin{bmatrix} \alpha x_2 \\ \beta x_1 \\ 0 \end{bmatrix} =: g_4(x) \\
[g_1, g_2] &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\mathcal{R}_1(x) = \operatorname{span}\{g_i(x), i = 1, \ldots, 4\}.$$

Step 3. If $\alpha = 0$ (i.e. $a_2 = a_3$), then $\mathcal{R}_1(x) = \mathcal{R}_0(x)$. So, $\mathcal{R}_c(x) = \mathcal{R}_0(x) = \operatorname{span}\{e_2, e_3\}$. If $\alpha \neq 0$, then $\mathcal{R}_1(x) \neq \mathcal{R}_0(x)$ and $\dim \mathcal{R}_1(x) = 2 < 3$ for $x_2 = x_3 = 0$. Hence, go back to Step 2.

Step 2-2.

$$\mathcal{R}_2(x) = \mathcal{R}_1(x) + \text{span}\{[f, g_i], [g_i, g_j],\ i, j = 1, 2, 3, 4\}$$

Since

$$[g_1, g_4] = \frac{\partial g_4}{\partial x} e_2 - \frac{\partial e_2}{\partial x} g_4(x) = \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix},\ (\alpha \neq 0)$$

$\mathcal{R}_2(x) = R^3$ (whole space).

Step 3-2 Since $\dim \mathcal{R}_2(x) = 3$ for any x, $R_c(x) = R^3$.

Therefore, if $a_2 \neq a_3$, then the system is locally strongly accessible from any point in R^3.

2 Stability for linear systems

We will discuss the stability of the linear system

$$\dot{x}(t) = Ax(t).$$

2.1 Asymptotic stability

The matrix A is stable (i.e., all the eigenvalues of A have negative real parts) if and only if for any $N < 0$, there exists a unique solution $P > 0$ for the Lyapunov equation

$$A^T P + PA = N.$$

In this case, if we define the Lyapunov function

$$V(x) := x^T P x,$$

with the solution P, then

$$\dot{V}(x(t)) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) = x^T(t)(A^T P + PA) x(t) < 0$$

for $x(t) \neq 0$.

Example

Check if the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is stable, without computing the eigenvalues.

Set $N = -I_2$ and solve the Lyapunov equation (you can use `lyap.m`):

$$\underbrace{\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} = - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{N}$$
$$\Leftrightarrow P = \frac{1}{4} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrix P is positive definite, and therefore, A is stable.

2.2 Critical stability

Suppose that the matrix A does not have eigenvalues with positive real part and has some eigenvalues on the imaginary axis. Such a case is called a *critical case*. In critical cases, the system is stable if and only if algebraic multiplicities of the eigenvalues on the imaginary axis equal geometric multiplicities.

Example

First, consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x,$$

where A has two eigenvalues at the origin (algebraic multiplicity is two). For the two eigenvalues, there is only one eigenvector $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ (geometric multiplicity is one). Hence, the system is *unstable*. Indeed,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_1(t) = x_{20}t + x_{10} \\ x_2(t) = x_{20} \end{cases}$$

and $x_2(t)$ diverges if $x_{20} \neq 0$.

Next, consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x,$$

where A has eigenvalues at $\pm i$, with algebraic multiplicity one. Since each eigenvalue corresponds to one eigenvector, algebraic and geometric multiplicities are the same for each eigenvalue. Hence, this system is *stable*. Indeed,

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{array} \right. \implies \left\{ \begin{array}{l} x_1(t) = r\sin(t+\phi) \\ x_2(t) = r\cos(t+\phi) \end{array} \right.$$

and the trajectory $\{(x_1(t), x_2(t))\}_t$ forms a circle with radius r.

3 Stability for nonlinear systems

3.1 Principle of stability in the first approximation

Consider a nonlinear system

$$\dot{x} = Ax + g(x),$$

where $g(x)$ indicates higher order terms than order one (i.e., g may include x_1^2, x_1x_2, x_2^2 etc.). Denote the set of all the eigenvalues of A by $\sigma(A)$. Then, $x = 0$ is

- **exponentially stable** if $\sigma(A) \subset C^-$. (C^- is the open left half-plane.)
- **unstable** if $\sigma(A) \cap C^+ \neq \emptyset$. ($C^+$ is the open right half-plane.)

If A has no eigenvalue in the open right half-plane but has at least one eigenvalue on the imaginary axis, then we need nonlinear stability theory, such as center manifold theory, to determine the stability of $x = 0$.

Next, consider a nonlinear system with a control

$$\dot{x} = Ax + g(x) + Bu,$$

where g is the same as above. If (A, B) is stabilizable, then $x = 0$ of the nonlinear system can be exponentially stable by using a state feedback $u = Fx$, where F is chosen so that $A + BF$ is stable.

Example

Consider a nonlinear system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}}_{A} x + g(x),$$

where g is a higher order term than order one.

If $\alpha = -1$ and $\beta = -2$: A has two eigenvalues at -1, and hence $x = 0$ is exponentially stable. (In fact, if α and β are negative, then A is a stable matrix and $x = 0$ is exponentially stable.)

If $\alpha = 0$ and $\beta = 1$: A has 1 eigenvalue at 1, and hence $x = 0$ is unstable.

If $\alpha = \beta = 0$: Since A has eigenvalues only on the imaginary axis, we cannot determine the stability of $x = 0$ by "Principle of stability in the first approximation".

3.2 Stability for a special but important nonlinear system

Consider a scalar nonlinear system

$$\dot{x} = ax^n, \ x(0) = x_0,$$

where a is a real constant and n is a positive integer. Study the stability of this system.

We consider several cases.

If $n = 1$: The system is linear.

$$x(t) = e^{at}x_0.$$

If $a < 0$: Since $x(t) \to 0$ as $t \to \infty$, $x = 0$ is *asymptotically stable.*

If $a = 0$: Since $x(t) = x_0$ for all t, $x = 0$ is *(critically) stable.*

If $a > 0$: Since $|x(t)| \to \infty$ as $t \to \infty$, $x = 0$ is *unstable.*

If $n > 1$: We solve the differential equation.

$$\begin{aligned} \dot{x} = ax^n \quad &\Rightarrow \quad \int x^{-n} dx = \int a dt \\ &\Rightarrow \quad \frac{1}{1-n} x^{1-n} = at + \frac{1}{1-n} x_0^{1-n}, \ \text{(since } x(0) = x_0\text{)} \\ &\Rightarrow \quad x(t)^{n-1} = \frac{1}{(1-n)at + x_0^{1-n}} \end{aligned}$$

If $a = 0$: Since $x(t) = x_0$ for all t, $x = 0$ is *critically stable.*

If $a \neq 0$: Since $\left|(1-n)at + x_0^{1-n}\right| \to \infty$ as $t \to \infty$, $|x(t)| \to 0$ as $t \to \infty$. **But** the question is if $|x(t)| \to \infty$ at some time $t_0 \in (0, \infty)$ for some x_0.

By setting the denominator of $x(t)^{n-1}$ equal zero,

$$t_0 := \frac{x_0^{1-n}}{a(n-1)} = \frac{1}{ax_0^{n-1}(n-1)}.$$

If $a < 0$ and n is odd: Since $x_0^{n-1} > 0$ for all nonzero x_0, $t_0 < 0$ and hence $|x(t)| \neq \infty$ and $x = 0$ is *asymptotically stable.*

Otherwise: We can always choose x_0 such that

$$t_0 = \frac{1}{ax_0^{n-1}(n-1)} > 0.$$

Thus, $x = 0$ is *unstable.*

In summary, $x = 0$ is

- **asymptotically stable** if $a < 0$ and n is odd,
- **critically stable** if $a = 0$,
- **unstable** otherwise.

Fact

The stability of the system

$$\dot{x} = ax^n + \mathcal{O}(|x|^{n+1})$$

is the same as the stability of $\dot{x} = ax^n$. (This fact will be useful when you learn center manifold theory.)