KTH Matematik

Exam March 8 2007 in 5B1873 Optimal Control.

Examiner: Ulf Jönsson, tel. 790 84 50.

Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All solutions and conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 12 Credits or more gives grade 3. 17 credits or more gives grade 4 and 22 or more credits gives grade 5.

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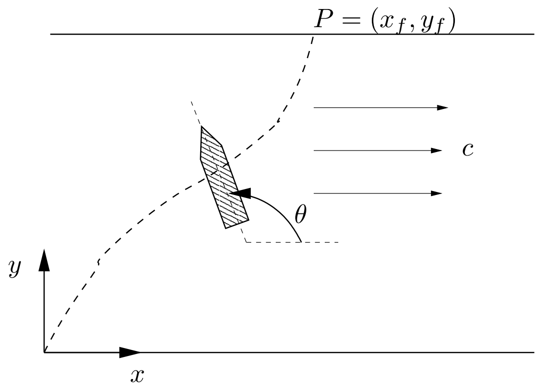
1. A boat crosses a river with a strong current c. The current makes the boat drift in x-diection, see Fig. 1. You would like to determine the steering angle such that the boat reaches the other side at a position $P = (x_f, y_f)$ with smallest possible x-coordinate. Assume that the boat starts in the origin $(x(0), y(0)) = (0, 0)$ and assume that its speed relative the water is v and thus $\dot{x}(t) = v\cos(\theta(t)) + c$ and $\dot{y}(t) = v\sin(\theta(t))$, where θ is the heading angle indicated in Fig 1.

(a) Formulate the problem as an optimal control problem on standard form. (2p)

(b) Prove that the optimal heading is constant. .... (2p)

(c) What is the optimal heading in the case when $c > v$? ....(1p)



Figur 1: A river boat crossing a river with heavy current c.

2. This problem consists of two questions.

(a) Determine which of the following partial differential equations $(a) - (d)$ corresponds to the following optimal control problem

$$\min_u x(1)^2 \quad \text{s.t} \quad \begin{cases} \dot{x} = 2u, & x(0) = x_0 \\ |u| \leq 1 \end{cases}$$

(a) $-V_t = -2V_x \text{sign}(V_x)$, $V(1,x) = x^2$
(b) $-V_t = -2V_x \text{sign}(V_x)$, $V(1,x) = 2x$
(c) $-V_t = -2V_x$, $V(1,x) = x^2$
(d) $-V_t = -\sin(V_x)(1+2V_x)$, $V(1,x) = x^2$

.... (3p)

(b) Consider the optimal control problem

$$J(x_0) = \min_u \int_0^1 f_{01}(x,u)dt + \int_1^2 f_{02}(x,u)dt$$
$$\text{s.t.} \quad \begin{cases} \dot{x} = f_1(x,u), & 0 \le t \le 1, \quad x(0) = x_0 \\ \dot{x} = f_2(x,u), & 1 \le t \le 2 \end{cases}$$

Below are two attempts of solving the problem. They cannot both be correct (and may both be wrong). Find and explain the error(s) in the reasoning. Is any of the two attempts correct?

Attempt 1:

$$\begin{aligned} J^*(x_0) = \; & \min_{u_1} \int_0^1 f_{01}(x_1,u_1)dt && + \; \min_{x_1(1)} \min_u \int_1^2 f_{02}(x_2,u_2)dt \\ & s.t.\ \dot{x}_1 = f_1(x_1,u_1),\ x_1(0) = x_0 && \quad s.t.\ \dot{x}_2 = f_2(x_2,u_2),\ x_2(1) = x_1(1) \\ = \; & \min_{x_1(1)} J_1^*(x_0) + J_2^*(x_1(1)) \end{aligned}$$

Attempt 2:

$$\begin{aligned} J^*(x_0) = \; & \min_u \{ \int_0^1 f_{01}(x,u)dt && + \; \min_u \int_1^2 f_{02}(x,u)dt \; \} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 && \quad s.t.\ \dot{x} = f_2(x,u), \end{aligned}$$

$$\begin{aligned} = \; & \min_u \left\{ \int_0^1 f_{01}(x,u)dt + J_2^*(x(1)) \right\} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 \end{aligned}$$

where

$$\begin{aligned} J_k^*(x_0) = \; & \min_u \int_{k-1}^k f_{0k}(x_k,u)dt \\ & s.t.\ \dot{x}_k = f_k(x_k,u),\ x_k(k-1) = x_0 \end{aligned}$$

.... (2p)

3. We will solve two similar optimal control problems.

(a) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) = 0,\ x(2) \in S_2 \end{cases}$$

where $S_2 = \{x \in \mathbf{R}^2 : x_2^2 - x_1 + 1 = 0\}$.

.... (2p) (*b*) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) \in S_0,\ x(2) \in S_2 \end{cases}$$

where $S_0 = \{x \in \mathbf{R}^2 : x_2^2 + x_1 = 0\}$ and S_2 is as above. .... (3p)

4. Consider the model predictive control algorithm

(a) Measure $x_{t|t} := x_t$.

(b) Determine $u_{t|t}$ by solving

$$\begin{aligned} &\min 2|x_{t+2|t}| + |u_{t+1|t}| + |u_{t|t}| \\ &\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = x_{t+k|t} + u_{t+k|t},\ k = 0, 1 \\ |u_{t+k|t}| \le 1,\ k = 0, 1 \end{cases} \end{aligned}$$

(c) Apply $u_t := u^*_{t|t}$

(d) Let $t := t + 1$ and go to 1.

Determine an explicit solution $u_{t|t} = \mu(x_{t|t})$ to this MPC problem.
.... (5p)

5. In this problem you will investigate an optimal control problem with autonomous dynamics. The constraint that the final state state is zero has certain implications that you will investigate.

(*a*) Consider the optimal control problem

$$\min \int_0^{t_f} (x^2 + u^2)dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = x + u,\ x(0) = x_0 \\ x(t_f) = 0,\ t_f > 0 \end{cases}$$

where the final time is a free variable to be optimized. If we apply dynamic programming then we get two possible feedback solutions

$$u = -(1 \pm \sqrt{2})x$$

Are both optimal?
.... (1p)

(*b*) Now consider the more general case

$$\min \int_0^{t_f} (\|x\|^2 + u^2)dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu,\ x(0) = x_0 \\ x(t_f) = 0,\ t_f > 0 \end{cases}$$

where (A, B) is a controllable pair. How do you compute the optimal solution? Justify your answer. .... (3p)

(c) Solve (b) when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

. . . . (1p)

Good luck!