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On smooth optimal control determination

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2004-02-20

Abstract

When using the Pontryagin Maximum Principle in optimal control problems, the most difficult part of the numerical solution is associated with the non-linear operation of the maximization of the Hamiltonian over the control variables. For a class of problems, the optimal control vector is a vector function with continuous time derivatives. A method is presented to find this smooth control without the maximization of the the Hamiltonian. Three illustrative examples are considered.

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Contents

The classical optimal control problem

- Consider the classical optimal control problem (OCP), Pontryagin *et al.* (1962), Lee and Marcus (1967), Athans and Falb (1966), etc.

$$\int_0^T f_0(x,u)dt \to \min \qquad (1)$$

$$\frac{dx}{dt} = f(x,u), \qquad (2)$$

$$x(0) = x_0,\ x(T) = x_T. \qquad (3)$$

- the control variables $u(t)\in\mathbf{R}^m$, the state variables $x(t)\in\mathbf{R}^n$, and $f(x,u)\in\mathbf{R}^n$ are column vectors, with $m\leq n$.
- $f_0(x,u)$, $f(x,u)$ are **smooth** in all arguments.
- The *Hamiltonian* is

$$H = p^T f(x,u) - f_0(x,u). \qquad (5)$$

where it holds for the column vector $p(t)\in\mathbf{R}^n$ of co-state variables, that

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \qquad (6)$$

according to the Pontryagin Maximum Principle (PMP).

Classical solution

$$\int_0^T f_0(x,u)dt \to \min \qquad (1)$$

$$\frac{dx}{dt} = f(x,u), \qquad (2)$$

$$x(0) = x_0,\ x(T) = x_T. \qquad (3)$$

$$H = p^T f(x,u) - f_0(x,u). \qquad (5)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \qquad (6)$$

- If an optimal solution (x^*,u^*) exists, then, by PMP, it holds that ***$H(x^*, u^*, p^*) \geq H(x^*, u, p^*)$*** implying here by smoothness, and the presence of constraint (3) only, that for $u = u^*$,

$$\frac{\partial H}{\partial u} = 0. \qquad (7)$$

or, with (5) inserted into (7),

$$p^T\frac{\partial f}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \qquad (8)$$

where $\partial f_0/\partial u$ is $1\times m$, and $\partial f/\partial u$ is $n\times m$.
- To find (x^*, u^*, p^*) the two point boundary value problem (2)-(6) must be solved.
- **At each *t*, (8) gives *u** as a function of *x* and *p*. (8) is often non-linear, and computationally costly**.
- $p(0)$ has as many unknowns as given end conditions $x(T)$.

The new idea without optimization w.r.t. *u*

$$\int_0^T f_0(x,u)dt \rightarrow \min \quad (1)$$

$$\frac{dx}{dt} = f(x,u), \quad (2)$$

$$x(0) = x_0,\; x(T) = x_T. \quad (3)$$

$$H = p^T f(x,u) - f_0(x,u). \quad (5)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \quad (6)$$

$$\frac{\partial H}{\partial u} = 0. \quad (7)$$

$$p^T \frac{\partial f}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \quad (8)$$

- **We note that (8) is linear in *p***.
- Assume that rank($\partial f/\partial u$)=m $\Rightarrow$ $\exists$ a non-singular $m\times m$ submatrix. Then, re-index the corresponding vectors

$$x = [x^a; x^b]; p = [p^a; p^b];\; f(x,u) = [f^a(x,u); f^b(x,u)]$$

where $\square^a$ denotes an m-vector. Then, (8) gives

$$p^{aT}\frac{\partial f^a}{\partial u} + p^{bT}\frac{\partial f^b}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \quad (9)$$

$$p^a = -\left(\frac{\partial f^a}{\partial u}^T\right)^{-1}\frac{\partial f^b}{\partial u}^T p^b + \left(\frac{\partial f^a}{\partial u}^T\right)^{-1}\frac{\partial f_0}{\partial u}^T \doteq A(x,p^b,u) \quad (10)$$

- Hence **by linear operations**, m elements of $p \in \mathbf{R}^n$, i.e. p^a, are computed as a function of u, x, and p^b.

The new idea, cont'd

$$\int_0^T f_0(x,u)dt \rightarrow \min \quad (1)$$

$$\frac{dx}{dt} = f(x,u), \quad (2)$$

$$x(0) = x_0,\; x(T) = x_T. \quad (3)$$

$$H = p^T f(x,u) - f_0(x,u). \quad (5)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \quad (6)$$

$$\frac{\partial H}{\partial u} = 0. \quad (7)$$

$$p^{aT}\frac{\partial f^a}{\partial u} + p^{bT}\frac{\partial f^b}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \quad (9)$$

$$p^a = \left(\frac{\partial f^{aT}}{\partial u}\right)^{-1}\left(\frac{\partial f_0^T}{\partial u} - \frac{\partial f^{bT}}{\partial u}p^b\right) \triangleq A(x,p^b,u) \quad (10)$$

- Differentiate (10):

$$\frac{dp^a}{dt} = \frac{\partial A}{\partial x}f(x,u) + \frac{\partial A}{\partial u}\frac{du}{dt} + \frac{\partial A}{\partial p^b}\frac{dp^b}{dt} \quad (11)$$

$$B \doteq \frac{\partial A}{\partial u}$$

- where B is assumed non-singular. (6) gives

$$\frac{dp^a}{dt} = -\frac{\partial H}{\partial x^a}^T = -\frac{\partial f^a}{\partial x^a}^T p^a - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T \quad (12)$$

$$\frac{dp^b}{dt} = -\frac{\partial H}{\partial x^b}^T = -\frac{\partial f^a}{\partial x^b}^T p^a - \frac{\partial f^b}{\partial x^b}^T p^b + \frac{\partial f_0}{\partial x^b}^T \doteq S(x,p^b,u) \quad (13)$$

- (10) into RHS of (12, 13), noting that dp^a/dt is given by the RHS of (11) and (12), and solving for du/dt, gives

$$\frac{du}{dt} = B^{-1}\left[-\frac{\partial f^a}{\partial x^a}^T A - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T - \frac{\partial A}{\partial x}f(x,u) - \frac{\partial A}{\partial p^b}\frac{dp^b}{dt}\right] \doteq F(x,p^b,u) \quad (14)$$

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Theorem

Theorem: If the optimal control problem (1)-(3), $m \le n$, has the optimal solution x^*, u^* such that u^* is smooth and belongs to the open set U, and if the Hamiltonian is given by (5), the Jacobians $\partial f^a/\partial u$ and $B = \partial p^a/\partial u$ are non-singular, then the optimal states x^*, co-states p^{*b}, and control u^* satisfy

$$\begin{aligned} \frac{du}{dt} &= F(x, p^b, u), \\ \frac{dx}{dt} &= f(x, u), \\ \frac{dp^b}{dt} &= S(x, p^b, u). \end{aligned} \qquad (15)$$

with the appropriate initial conditions $u(0)=u_0$, $p^b(0)=p^b_0$ to be found.

$$\int_0^T f_0(x, u)dt \to \min \qquad (1)$$

$$\frac{dx}{dt} = f(x, u), \qquad (2)$$

$$x(0) = x_0, \; x(T) = x_T. \qquad (3)$$

$$H = p^T f(x, u) - f_0(x, u). \qquad (5)$$

Remark: if $m=n$, then $x^a=x$, and $p^a=p$, and (15) becomes

$$\begin{aligned} \frac{d\mathbf{x}}{d\mathbf{t}} &= \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \frac{d\mathbf{u}}{d\mathbf{t}} &= \mathbf{F}(\mathbf{x},\mathbf{u}) \end{aligned} \qquad (15')$$

Remark: The number of equations in (15) is $2n$, just as in PMP, but without the maximization of the Hamiltonian.

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Example 1: Rigid body rotation

Stopping axisymmetric rigid body rotation (Athans and Falb, 1963)

$$\begin{aligned} \frac{dx}{dt} &= ay + u_1, \\ \frac{dy}{dt} &= -ax + u_2 \end{aligned} \qquad (16)$$

$$x(T) = 0, y(T) = 0 \qquad (17)$$

$$J = \frac{1}{4}\int_0^T (u_1^2 + u_2^2)^2 dt \to \min$$

$$p = A = [\begin{array}{c} u_1(u_1^2 + u_2^2) \\ u_2(u_1^2 + u_2^2) \end{array}]$$

$$\begin{aligned} \frac{dp_x}{dt} &= ap_y \\ \frac{dp_y}{dt} &= -ap_x \end{aligned}$$

$$B = \left(\begin{array}{cc} 3u_1^2 + u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_1^2 + 3u_2^2 \end{array}\right)$$

$$B^{-1} = \frac{1}{(3[u_1^2 + u_2^2]^2)}\left(\begin{array}{cc} u_1^2 + 3u_2^2 & -2u_1u_2 \\ -2u_1u_2 & 3u_1^2 + u2^2 \end{array}\right)$$

$$\begin{aligned} \frac{du_1}{dt} &= au_2, \\ \frac{du_2}{dt} &= -au_1 \end{aligned} \qquad (23)$$

$$H = p^T f(x, u) - f_0(x, u). \qquad (5)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \qquad (6)$$

$$\mathbf{p^a} = \left(\frac{\partial \mathbf{f^a}^T}{\partial \mathbf{u}}\right)^{-1}\left(\frac{\partial \mathbf{f_0}^T}{\partial \mathbf{u}} - \frac{\partial \mathbf{f^b}^T}{\partial \mathbf{u}}\mathbf{p^b}\right) \triangleq \mathbf{A}(\mathbf{x}, \mathbf{p^b}, \mathbf{u}) \qquad (10)$$

$$B \doteq \frac{\partial A}{\partial u}$$

$$\begin{aligned} \frac{d\mathbf{x}}{d\mathbf{t}} &= \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \frac{d\mathbf{u}}{d\mathbf{t}} &= \mathbf{F}(\mathbf{x},\mathbf{u}) \end{aligned} \qquad (15')$$

$$\frac{du}{dt} = B^{-1}[-\frac{\partial f^a}{\partial x^a}^T A - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T - \frac{\partial A}{\partial x} f(x, u) - \frac{\partial A}{\partial p^b}\frac{dp^b}{dt}] \doteq F(x, p^b, u) \quad (14)$$

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Example 1: Rigid body rotation, cont'd

$$\frac{dx}{dt} = ay + u_1, \quad \frac{dy}{dt} = -ax + u_2 \quad (16)$$

$$x(T) = 0, y(T) = 0 \quad (17)$$

$$\frac{du_1}{dt} = au_2, \quad \frac{du_2}{dt} = -au_1 \quad (23)$$

- (23) ⇒

$$u_1^2 + u_2^2 = C^2$$

- Polar coordinates:

$$x = r \sin\theta, \quad y = r\cos\theta \quad (24)$$

- then, clearly,

$$u_1 = -C\frac{x}{\sqrt{x^2+y^2}}, \quad u_2 = -C\frac{y}{\sqrt{x^2+y^2}} \quad (25)$$

$$\frac{dr}{dt} = -C, \quad \frac{d\theta}{dt} = a. \quad (26)$$

⇒ ($u1$, $u2$) and (x, y) rotate collinearly with the same angular velocity a.

- Let

$$\sqrt{x(0)^2+y(0)^2} = R$$

then, from (17), (25), (26),

$$u_1(0)/C = -x(0)/R$$
$$u_2(0)/C = -y(0)/R$$
$$C = R/T$$

The problem is solved without maximizing the Hamiltonian!

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Example 2: Optimal spacing for greenhouse lettuce growth

Optimal variable spacing policy (Seginer, Ioslovich, Gutman), assuming constant climate

$$\frac{dv}{dt} = \frac{v}{W}G(W), \quad J = \int_0^T \frac{v}{W}c_R\,dt \quad (30)$$

$$v(T) = v_T \quad (31)$$

$$v = aW$$

with v [kg/plant] = dry mass, G [kg/m^2/s] net photosynthesis, W [kg/m^2] plant density (control), a [m^2/plant] spacing, v_T marketable plant mass, and final time T [s] free.

$$H = \frac{v}{W}(pG(W) - c_R) \quad (32)$$

$$\frac{\partial H}{\partial W} = -\frac{v}{W^2}(pG(W) - c_R) + \frac{vp}{W}\frac{\partial G}{\partial W}$$

- p is obtained from $\partial H/\partial W = 0$,

$$p = \frac{c_R}{G(W) - W\frac{\partial G}{\partial W}} \quad (34)$$

- $$\frac{dp}{dt} = -\frac{\partial H}{\partial v} = -\frac{pG(W) - c_R}{W} \quad (35)$$
- (34), (35) ⇒

$$\frac{dp}{dt} = -\frac{c_R\frac{\partial G}{\partial W}}{(G - W\frac{\partial G}{\partial W})} \quad (36)$$

- Differentiating (34),

$$\frac{dp}{dt} = \frac{dW}{dt}\frac{c_R W\frac{\partial^2 G}{\partial W^2}}{(G - W\frac{\partial G}{\partial W})^2} \quad (37)$$

- (36), (37) ⇒

$$\frac{dW}{dt} = -\frac{\frac{\partial G}{\partial W}(G - W\frac{\partial G}{\partial W})}{W\frac{\partial^2 G}{\partial W^2}} \quad (38)$$

- Free final time ⇒

$$H(T) = 0 \quad (39)$$

- Then, at $t=T$, (39,32,34):

$$v\frac{c_R\frac{\partial G}{\partial W}}{(G - W\frac{\partial G}{\partial W})} = 0 \quad (40)$$

- (38, 40) show that ∀ t, W^* satisfies

$$\partial G(W^*)/\partial W^* = 0$$

- For free final time, (38)... also w/o maximization of the Hamiltonian, I&G 99

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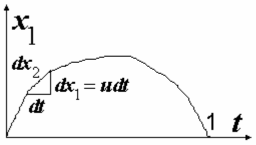
Example 3: Maximal area under a curve of given length

Variational, isoparametric problem, e.g. Gelfand and Fomin, (1969).



$$\int_0^1 x_1 dt \to \max \quad (42)$$

$$\frac{dx_1}{dt} = u, \quad (43)$$

$$\frac{dx_2}{dt} = \sqrt{1+u^2}. \quad (44)$$

$$x_1(0) = x_1(1) = 0$$

$$x_2(0) = 0,\ x_2(1) = \pi/3 \quad (45)$$

$$H = p_1 u + p_2\sqrt{1+u^2} + x_1 \quad (46)$$

$$\frac{dp_1}{dt} = -1, \quad \frac{dp_2}{dt} = 0. \quad (47)$$

$$\frac{\partial H}{\partial u} = p_1 + p_2\frac{u}{\sqrt{1+u^2}} = 0 \quad (48)$$

- Here, it is possible to solve (48) for u, but let us solve for p_1

$$p_1 = -p_2\frac{u}{\sqrt{1+u^2}} \quad (49)$$

- Differentiating (49), and using (47) gives

$$\frac{du}{dt} = \frac{(1+u^2)^{3/2}}{p_2} \quad (50)$$

- Guessing p_2=constant and $u(0)$, and integrate (43), (44), (50), such that (45) is satisfied, yields

$$p_2 = -1 \qquad u(0) = 1/\sqrt{(3)} \qquad !$$

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Conclusions

- A method to find the smooth optimal control for a class of optimal control problems was presented.
- The method does not require the maximization of the Hamiltonian over the control.
- Instead, the ODEs for *m* co-states are substituted for ODEs for the *m* smooth control variables.
- Three illustrative examples were given.