KTH Mathematics

SF2812 Applied linear optimization, final exam
Saturday February 18 2012 9.00–14.00
Brief solutions

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1. (a) The dual (DLP_δ) of (LP_δ) may be written as

$$(DLP_\delta) \qquad \begin{array}{ll} \max & (b+\delta e_3)^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

The given y and s are feasible to (DLP_δ), so insertion of the given y gives and underestimate of the optimal value of (LP_δ) as $63 + 6\delta$. This underestimate is exact as long as y and s are optimal to (DLP_δ).

(b) The optimality of the given y and s corresponds to the corresponding basic variables being nonnegative. This gives $x_B = B^{-1}(b+\delta e_3) \geq 0$. Insertion gives

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e., $-1/2 \leq \delta \leq 2$. Hence, the underestimate is exact for $-1/2 \leq \delta \leq 2$.

2. (See the course material.)

3. (a) For a fix vector $u \in I\!R^n$, Lagrangian relaxation of the first group of constraints gives

$$\begin{array}{rll} \varphi(u) = & \text{minimize} & \displaystyle\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{j=1}^n f_j z_j - \sum_{i=1}^n u_i \left(\sum_{j=1}^n x_{ij} - 1 \right) \\ & \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \geq b_j z_j, \quad j = 1,\dots,n, \\ & & x_{ij} \in \{0,1\},\ z_j \in \{0,1\}, \quad i = 1,\dots,n,\ j = 1,\dots,n, \end{array}$$

This problem decomposes into one problem for each j as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n (c_{ij} - u_i) x_{ij} - f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \geq b_j z_j, \\ & x_{ij} \in \{0,1\},\ z_j \in \{0,1\}, \quad i = 1,\dots,n, \end{array}$$

for $j = 1, \ldots, n$. For each j, we may solve two problems by equating $z_j = 0$ and $z_j = 1$ respectively. For $z_j = 0$ we obtain $x_{ij} = 0$ or $x_{ij} = 1$ depending on whether $c_{ij} - u_i$ is positive or negative. For $z_j = 1$ we obtain a "knapsack-like" problem in the x_{ij}-variables, $i = 1, \ldots, n$.

(b) If these constraints are relaxed, the resulting Lagrangian relaxation problem has the integrality property, i.e., the extreme points are integer valued. Thus, the dual problem gives the same bound as the linear programming relaxation. We would thus expect the relaxation of the previous exercise to give a tighter bound.

4. As $W/w_1 = 4\frac{2}{3}$, the cut patterns with w_1-rolls only is given by $(4\ 0\ 0)^T$. The two other analogous cut patterns are given by $(0\ 2\ 0)^T$ and $(0\ 0\ 2)^T$.

Consequently, we obtain $A_1 = (4\ 0\ 0)^T$, $A_2 = (0\ 2\ 0)^T$ and $A_3 = (0\ 0\ 2)^T$, so that

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 10 \\ 45 \\ 25 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enter the basis.

We obtain the subproblem

$$\begin{array}{rll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 7\alpha_3 \leq 14, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal solutions to the subproblem are given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 0$, and $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 0$, with optimal value -1/4. As suggested in the statement, we let $A_4 = (1\ 2\ 0)^T$ with

$$p_B = -B^{-1}A_4 = \begin{pmatrix} -\frac{1}{4} \\ -1 \\ 0 \end{pmatrix}.$$

The minimum ratio occurs for $\alpha_{\max} = 40$, when the first basic variable becomes zero, so that x_1 leaves the basis. Hence,

$$B = \begin{pmatrix} A_4 & A_2 & A_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so that

$$x_B = \begin{pmatrix} x_4 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 40 \\ 5 \\ 25 \end{pmatrix}, \quad y = y = B^{-T}e = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enter the basis.

We obtain the subproblem

$$\begin{array}{rll} 1 - & \frac{1}{2}\text{maximize} & \alpha_2 + \alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 7\alpha_3 \leq 14, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is zero. Hence, the linear program has been solved.

However, as it so happens that x_B is integer valued, the original problem has been solved as well. An optimal solution to the original problem is thus given by cutting 40 W-rolls according to cut pattern $(1\ 2\ 0)^T$, 5 W-rolls according to cut pattern $(0\ 2\ 0)^T$ and 25 rolls according to cut pattern $(0\ 0\ 2)^T$.

(Note that this is very special. In general x_B will not take on integer values.)

5. (a) There are four active constraints at $\bar{x}$, all constraints except $x_3 \geq 0$. Hence, as $n = 3$, there are more than n active constraints, i.e., the point is degenerate.

(b) The point $x(t) = (-t\ \ -t\ 1)^T$, for $t > 0$, satisfies $a_i^T x(t) \geq b_i$, $i = 2, \ldots, 5$. However, as $a_1^T x(t) = -2t - 2 < -2 = b_1$, $x(t) \notin P$. Hence, $a_1^T x \geq b_1$ is not redundant. We may analogously consider $(-t\ t\ 1)^T$, $(t\ \ -t\ 1)^T$, and $(t\ t\ 1)^T$ for constraints 2, 3 and 4. Finally, the last constraint is not redundant, as $(0\ \ 0\ \ -1)^T$ satisfies all constraints except number 5. Hence, the description of P given by A and b contains no redundant constraints.

(c) A straightforward convergence proof implies strictly decreasing objective function value. This means a nonzero step. At a degenerate vertex, the simplex method utilizes n linearly independent constraints to form the basis matrix. As the vertex is degenerate, there are additional active constraints, so that the steplength may be zero. Hence, it may not be assumed that the steplength is nonzero and such a convergence proof would not hold.