SF2822 Applied nonlinear optimization, final exam
Thursday May 31 2018 8.00–13.00
Brief solutions

1. (a) This claim is true.

The output of "`active=find(g<sqrt(eps))`" shows that constraints 12, 17 and 21 are active at x^*.

In addition, the comand "`rank(A(active,:))`" gives 3, showing that the constraint gradients of the active constraints are linearly independents, i.e., x^* is a regular point.

(b) This claim is true.

The output of "`norm(gradf-A'*lambdastar)`" is of the order of machine precision, i.e., $\nabla f(x^*) - A(x^*)^T\lambda^*$ is numerically zero.

In addition, the output of "`[g lambdastar]`" shows that $g(x^*) \geq 0$, $\lambda^* \geq 0$ and $g_i(x^*)\lambda_i^* = 0$, $i = 1, \ldots, 24$. Consequently, x^* together with λ^* satisfy the first-order necessary optimality conditions.

(c) This claim is false.

The output of "`eig(Z'*HessL*Z)`" shows that $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ has two negative eigenvalues, hence contradicting the required positive semidefiniteness, where Z is a matrix whose columns form a basis for the nullspace of the Jacobian of the active constraints.

(d) This claim is false.

The second-order sufficient optimality conditions require $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ positive definite, and they cannot hold since the second-order necessary conditions do not hold.

(e) This claim is false.

Since x^* is a regular point at which the second-order necessary conditions do not hold, it cannot be a local minimizer. Therefore, it cannot be a global minimizer.

2. No constraint is in the working set at the initial point, i.e., $\mathcal{W} = \emptyset$. With $H = I$ and $c = 0$ we obtain

$$Hp^{(0)} = -(Hx^{(0)} + c).$$

Insertion of numeric values gives

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1^{(0)} \\ p_2^{(0)} \end{pmatrix} = -\begin{pmatrix} 15 \\ 3 \end{pmatrix}.$$

We obtain

$$p^{(0)} = \begin{pmatrix} -15 & -3 \end{pmatrix}^T, \quad \lambda^{(1)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{1}{3},$$

where the minimium is attained for $i = 3$. Consequently, $\alpha^{(0)} = 1/3$ so that

$$x^{(1)} = x^{(0)} + \alpha^{(0)} p^{(0)} = \begin{pmatrix} 15 \\ 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -15 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix},$$

with $\mathcal{W} = \{3\}$. The solution to the corresponding equality-constrained quadratic progam is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ -\lambda_3^{(2)} \end{pmatrix} = -\begin{pmatrix} 10 \\ 2 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(1)} = \begin{pmatrix} -10 & 0 \end{pmatrix}^T, \quad \lambda^{(2)} = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(1)} < 0} \frac{a_i^T x^{(1)} - b_i}{-a_i^T p^{(1)}} = \frac{3}{5},$$

where the minimium is attained for $i = 1$. Consequently, $\alpha^{(1)} = 3/5$ so that

$$x^{(2)} = x^{(1)} + \alpha^{(1)} p^{(1)} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} -10 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$

with $\mathcal{W} = \{1, 3\}$. The solution to the corresponding equality-constrained quadratic progam is given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ -\lambda_1^{(3)} \\ -\lambda_3^{(3)} \end{pmatrix} = -\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(2)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} 4 & 0 & -2 \end{pmatrix}^T.$$

As $p^{(2)} = 0$, it follows that $x^{(3)} = x^{(2)}$ and the corresponding equality-constrained problem has been solved. However, since $\lambda_3^{(3)} < 0$, constraint 3 is deleted so that $\mathcal{W} = \{1\}$. The solution to the corresponding equality-constrained quadratic progam is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(3)} \\ p_2^{(3)} \\ -\lambda_1^{(4)} \end{pmatrix} = -\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(3)} = \begin{pmatrix} -1 & 1 \end{pmatrix}^T, \quad \lambda^{(4)} = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(3)} < 0} \frac{a_i^T x^{(3)} - b_i}{-a_i^T p^{(3)}} = 4,$$

where the minimium is attained for $i = 2$. Since $\alpha_{\max} > 1$, we let $\alpha^{(3)} = 1$ so that

$$x^{(4)} = x^{(3)} + p^{(3)} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since $\lambda^{(4)} \geq 0$, the optimal solution has been found.

3. (See the course material.)

4. (a) In an interior method, we need to ensure that the constraint, on which the barrier transformation is applied, is satisfied with strict inequality. Hence, if the barrier is applied on $g(x) \geq 0$, we must that $g(x^{(k)}) > 0$ for all iterates k. Since $g(x^{(0)}) = -2 < 0$, some reformulation is needed.

(b) The Newton step Δx, Δs, $\Delta\lambda$ is given by

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x,\lambda) & 0 & -A(x)^T \\ A(x) & -I & 0 \\ 0 & \Lambda & S \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x) - A(x)^T\lambda \\ g(x) - s \\ S\lambda - \mu e \end{pmatrix},$$

where the matrix and righ-hand side is evaluated at the particular iterate x, s, λ.

In our case we get

$$\mathcal{L}(x,\lambda) = \tfrac{1}{2}x_1^2 + x_2 - \lambda(-x_1 - x_2^2 + 1),$$

so that

$$\nabla^2_{xx}\mathcal{L}(x,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 2\lambda \end{pmatrix}, \qquad \nabla f(x) = \begin{pmatrix} x_1 \\ 1 \end{pmatrix},$$
$$A(x) = \begin{pmatrix} -1 & -2x_2 \end{pmatrix}, \qquad g(x) = -x_1 - x_2^2 + 1.$$

The Newton system then becomes

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2\lambda & 0 & 2x_2 \\ -1 & -2x_2 & -1 & 0 \\ 0 & 0 & \lambda & s \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} x_1 + \lambda \\ 1 + 2x_2\lambda \\ -x_1 - x_2^2 + 1 - s \\ s\lambda - \mu \end{pmatrix}.$$

The initial value of s should be strictly positive. For example, let $s^{(0)} = 1/2$. Then, for the first iteration we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ -1 & -2 & -1 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ \Delta s^{(0)} \\ \Delta\lambda^{(0)} \end{pmatrix} = - \begin{pmatrix} 4 \\ 5 \\ -\frac{5}{2} \\ 0 \end{pmatrix}.$$

(c) The next iterate is given by $x^{(1)} = x^{(0)} + \alpha^{(0)}\Delta x_1^{(0)}$, $s^{(1)} = s^{(0)} + \alpha^{(0)}\Delta s^{(0)}$, $\lambda^{(1)} = \lambda^{(0)} + \alpha^{(0)}\Delta\lambda^{(0)}$, where $\alpha^{(0)}$ is given by some approximate linesearch. The steplength $\alpha^{(0)}$ must be chosen such that $s^{(0)} + \alpha^{(0)}\Delta s^{(0)} > 0$ and $\lambda^{(0)} + \alpha^{(0)}\Delta\lambda^{(0)} > 0$.

5. (a) The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & A(x^{(0)})p \geq -g(x^{(0)}). \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & 2p_1^2 + 2p_2^2 - p_2 \\ \text{subject to} & 2p_1 - 2p_2 \geq 1, \\ & -2p_1 - 2p_2 \geq 1. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem. The quadratic program is convex. By symmetry, we guess that $p_1^{(0)} = 0$, which gives $p_2^{(0)} = -1/2$. If this is correct, there must exist nonnegative Lagrange multipliers $\lambda^{(1)}$ such that

$$\begin{pmatrix} 4p_1^{(0)} \\ 4p_2^{(0)} - 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1^{(0)} \\ \lambda_2^{(0)} \end{pmatrix}.$$

There is a unique solution given by $\lambda^{(1)} = (3/4 \ 3/4)^T$, which is nonnegative. Our guess was therefore correct. Finally, $x^{(1)} = x^{(0)} + p^{(0)} = (0 \ 1/2)^T$.

(b) Linearization of the objective function in (NLP') gives

$$f(x^{(k)} + p) + Me^T(u^{(k)} + q) \approx f(x^{(k)}) + \nabla f(x^{(k)})^T p + Me^T u^{(k)} + Me^T q.$$

Since u only appears linearly in (NLP'), the quadratic part of the objective function in (QP') is identical to that in the objective function of the QP subproblem associated with (NLP). The objective function therefore becomes

$$\tfrac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(k)},\lambda^{(k)})p + \nabla f(x^{(k)})^T p + Me^T q.$$

Setting a linearization of the constraints feasible in (NLP') gives

$$\begin{aligned} g(x^{(k)} + p) + u^{(k)} + q \approx g(x^{(k)}) + A(x^{(k)}p + u^{(k)} + q &\geq 0, \\ u^{(k)} + q &\geq 0. \end{aligned}$$

The QP subproblem associated with (NLP') at iteration k may consequently be written as

$$(QP') \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(k)},\lambda^{(k)})p + \nabla f(x^{(k)})^T p + Me^T q \\ \text{subject to} & A(x^{(k)})p + q \geq -g(x^{(k)}) - u^{(k)}, \\ & q \geq -u^{(k)}. \end{array}$$

If in addition $u^{(k)} = 0$, (QP') takes the form

$$(QP') \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}p^T \nabla^2_{xx}\mathcal{L}(x^{(k)}, \lambda^{(k)})p + \nabla f(x^{(k)})^T p + Me^T q \\ \text{subject to} & A(x^{(k)})p + q \geq -g(x^{(k)}), \\ & q \geq 0. \end{array}$$

The first-order necessary optimality conditions for (QP') then become

$$\begin{aligned} \nabla^2_{xx}\mathcal{L}(x^{(k)}, \lambda^{(k)})p + \nabla f(x^{(k)}) &= A(x^{(k)})^T\lambda, \\ Me &= \lambda + \eta, \\ A(x^{(k)})p + q &\geq -g(x^{(k)}), \\ \lambda &\geq 0, \\ (A(x^{(k)})p + q + g(x^{(k)}))^T\lambda &= 0, \\ q &\geq 0, \\ \eta &\geq 0, \\ q^T\eta &= 0. \end{aligned}$$

Now assume that $Me - \lambda > 0$. Then, $\eta > 0$, since $\eta = Me - \lambda$. But then, the complementarity condition $q^T\eta = 0$ and nonnegativity of q gives $q = 0$. Then, the optimality conditions take the form

$$\begin{aligned} \nabla^2_{xx}\mathcal{L}(x^{(k)}, \lambda^{(k)})p + \nabla f(x^{(k)}) &= A(x^{(k)})^T\lambda, \\ A(x^{(k)})p &\geq -g(x^{(k)}), \\ \lambda &\geq 0, \\ (A(x^{(k)})p + g(x^{(k)}))^T\lambda &= 0, \end{aligned}$$

which are exactly the optimality conditions of the QP subproblem associated with (NLP). Therefore, based on the optimality conditions we conclude that $q^{(k)} = 0$ in addition to $p^{(k)}$ and $\lambda^{(k+1)}$ being optimal solution and Lagrange multipliers respectively of the corresponding QP subproblem associated with (NLP).