CHAPTER 2

Invariant and controlled invariant subspaces

In this chapter we introduce two important concepts: invariant subspace and controlled invariant subspace, which will be used later on to solve many control problems.

2.1. Invariant subspaces

Consider an $n$-dimensional linear system

\[ \dot{x} = Ax \]

where $x \in \mathbb{R}^n$.

**Definition 2.1.** A set $\Omega \subseteq \mathbb{R}^n$ is called an invariant set of (2.1) if for any initial condition $x_0 \in \Omega$, we have $x(x_0, t) = e^{At}x_0 \in \Omega$, $\forall t \geq 0$.

Some trivial examples of invariant sets are $\mathbb{R}^n$ and $x = \{0\}$.

In this course we only consider a special class of invariant sets: invariant subspaces. Now let us discuss conditions for a subspace $S$ to be invariant.

Since by Taylor expansion we have

\[ x(x_0, t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots, \]

it is obvious that if $A^i x_0 \in S \forall i \geq 0$, then $x(x_0, t) \in S$, $\forall t \geq 0$. Naturally this argument is true only if $S$ is a linear subspace. It is easy to see as a sufficient condition

\[ A^i z \in S \forall z \in S. \]

In other words, this condition implies that if we define a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$: $w = Az$, then the image of $S \subseteq \mathbb{R}^n$ is contained in $S$. We denote this by

\[ AS \subseteq S. \]

Now we show this condition is also necessary for $S$ to be invariant.

**Proposition 2.1.** A necessary and sufficient condition for a linear subspace $S$ to be invariant under (2.1) is that condition (2.3) holds.

**Proof**

We only show the necessity here. Suppose there exists a point $x_0 \in S$ such that $Ax_0 \notin S$. Then when $t$ is sufficiently small, we have

\[ x(x_0, t) = x_0 + tAx_0 + O(t^2), \]

but $Ax_0 \notin S$. Therefore, $x(x_0, t) \notin S$ for some $t > 0$. This contradicts the assumption that $S$ is invariant.
which does not belong to $S$, since $S$ is closed.

**Example 2.1.** Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$  

Show that $S = \text{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ is invariant.

We first use the definition of invariant set to show this. It is easy to see that the set can be redefined as $S = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. Then to show $S$ to be invariant is to show $x_1(x_0, t) + x_2(x_0, t) = 0 \forall t \geq 0$ if $x_0 \in S$. This is equivalent to showing $\dot{x}_1 + \dot{x}_2 = 0$ for all $(x_1, x_2)^T \in S$. We have $\dot{x}_1 + \dot{x}_2 = -2x_2 - x_2 + x_1 = -(x_1 + x_2) = 0$ if $(x_1, x_2)^T \in S$.

We can also show this with Proposition 2.1, since $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

**Remark 2.1.** As an example to show the above result is only true for subspaces, we consider a circle defined by $R = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. It is easy to show (as an exercise) that this set is invariant under the system

$$\begin{align*}
\dot{x}_1 &= \omega x_2 \\
\dot{x}_2 &= -\omega x_1,
\end{align*}$$

where $\omega$ is any positive number.

We ask the reader to check if $Az \subset R$ for any $z \in R$.

Then we can use condition (2.3) as an alternative definition for invariant subspace.

**Definition 2.2.** A linear subspace $S$ is $A$-invariant (invariant under $\dot{x} =Ax$) iff $AS \subseteq S$.

### 2.2. Controlled invariant subspaces

Now we consider a control system

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

**Definition 2.3.** $S$ is called a controlled invariant subspace of (2.4) if there exists a feedback control $u = Fx$ such that $S$ is an invariant subspace of

$$\dot{x} = (A + BF)x.$$

Similar to invariant subspace, we can also give another equivalent definition.

**Definition 2.4.** $S$ is an $(A, B)$-invariant (controlled invariant) subspace if there exists a matrix $F$ such that

$$\dot{x} = (A + BF)x.$$

Such an $F$ is called a **friend** of $S$. 
We denote the set of friends by $F(S)$. The following theorem provides a fundamental characterization of $(A,B)$-invariant subspaces that removes the explicit involvement of the feedback matrix $F$.

**Theorem 2.2.** $S$ is $(A,B)$-invariant if and only if

\[
AS \subseteq S + \text{Im } B.
\]

**Proof**

Necessity: Suppose $F$ is a friend, then

\[
(A + BF)S \subseteq S.
\]

or

\[
AS \subseteq S - B(FS).
\]

Since $B(FS) \subseteq \text{Im } B$, thus (2.6) holds.

Sufficiency: The proof is constructive and is given as follows.

We now give an algorithm for finding a friend of $V$, which also serves as a proof of the sufficiency of Theorem 2.2.

**Algorithm for finding $F$**

Let \( \{v_1, v_2, \ldots, v_r\} \) be a basis for $V$. Since $V$ satisfies $AV \subseteq V + \text{Im } B$, there is for each $i = 1, \ldots, r$ a $w_i \in V$ and a $u_i \in \mathbb{R}^m$ such that

\[
Av_i = w_i + Bu_i.
\]

Let $F$ be a $m \times n$-matrix such that $Fv_i = -u_i$ for $i = 1, 2, \ldots, r$ (if $r < n$ then $F$ is not unique). Then $Av_i = w_i - BFv_i$, i.e., $(A + BF)v_i = w_i \in V$ and therefore $(A + BF)V \subseteq V$.

**Example 2.2.** Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the subspace $V = \{x \in \mathbb{R}^2 : x_1 = x_2\}$ $(A,B)$-invariant? If so, find a friend of $V$.

Clearly, $V$ is spanned by $v = [1 \ 1]'$. Since

\[
Av = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V + \text{Im } B,
\]

$V$ is $(A,B)$-invariant, and we can let $u = 2$. To find $F$ we must solve the under-determined system of equations $Fv = -u$, i.e.,

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2.
\]

The set $F(V)$ is the affine space \( \{(\lambda - 2, -\lambda) : \lambda \in \mathbb{R}\} \). Choose, e.g., $F = \begin{bmatrix} -2 & 0 \end{bmatrix}$.

Is the subspace $V := \{x \in \mathbb{R}^2 : x_1 = 0\}$ $(A,B)$-invariant?
2.3. Reachability subspaces

In the rest of this chapter, we study the most elementary class of controlled invariant subspaces: reachability (controllability) subspace.

**Definition 2.5.** We use the notation $\langle A|S \rangle$ to denote the minimal $A$-invariant subspace that contains subspace $S$.

Naturally if $S$ is already $A$-invariant, then $\langle A|S \rangle = S$. $\langle A|S \rangle$ can be computed in the following way:

1. Let $S_0 = S$, check if $AS_0 \subseteq S_0$. If yes, stop. Otherwise,
2. Let $S_{k+1} = AS_k + S_k$, $k \geq 0$.
3. Check if $AS_{k+1} \subseteq S_{k+1}$. If yes, stop. Otherwise return to step 2.

Consider again (2.4). Recall that the reachable (controllable) subspace of (2.4) can be defined with our notation as
$$\langle A|\text{Im } B \rangle = \text{span}\{B, AB, \cdots, A^{n-1}B\},$$

namely, the minimal $A$-invariant subspace that contains $\text{Im } B$. However, for many complex control problems, such as the problem of controllability under constraints discussed in the introduction, more refined study of reachability is needed.

Now consider the feedback law
$$u = Fx + Gv.$$  

The corresponding closed-loop system
$$\dot{x} = (A + BF)x + BGv$$

has the reachable subspace
$$\mathcal{R} = \langle A + BF|\text{Im } BG \rangle.$$

**Remark 2.2.** By construction, $\mathcal{R}$ is $(A, B)$-invariant.

**Definition 2.6.** A subspace $\mathcal{R}$ is called a reachability subspace of (2.4) if there are $F$ and $G$ such that (2.8) holds.

**Example 2.3.** If $G = I$ then
$$\mathcal{R} = \langle A + BF|\text{Im } B \rangle = \langle A|\text{Im } B \rangle,$$

is the reachable subspace. If $G = 0$ then $\mathcal{R} = 0$. For a SISO-system it is obvious that these are the only possible reachability subspaces.

We now proceed with the analysis of reachability subspaces. The first theorem shows that the matrix $G$ can be removed from the characterization of $\mathcal{R}$ at the price of an implicit characterization, which however is of great use.

**Theorem 2.3.** A subspace $\mathcal{R}$ is a reachability subspace if and only if there is an $F$ such that
$$\mathcal{R} = \langle A + BF|\text{Im } B \cap \mathcal{R} \rangle.$$
Proof

Necessity: Suppose $\mathcal{R}$ is a reachability subspace, i.e.,

\[(2.10) \quad \mathcal{R} = \langle A + BF | \text{Im } BG \rangle\]

for some $F$ and $G$. Then $\text{Im } BG \subseteq \mathcal{R}$ and $\text{Im } BG \subseteq \text{Im } B$, i.e.,

$$\text{Im } BG \subseteq \text{Im } B \cap \mathcal{R}.$$ 

Hence,

\[(2.11) \quad \mathcal{R} \subseteq \langle A + BF | \text{Im } B \cap \mathcal{R} \rangle.\]

But $\mathcal{R}$ is $(A, B)$-invariant and therefore

\[(A + BF)^k \mathcal{R} \subseteq \mathcal{R} \quad \text{for } k \geq 1\]

and

\[(2.12) \quad \langle A + BF | \text{Im } B \cap \mathcal{R} \rangle \subseteq \mathcal{R}.\]

Now (2.9) follows from (2.11) and (2.12).

Sufficiency: Suppose that (2.9) holds. It is enough to show that there is a $G$ such that $\text{Im } B \cap \mathcal{R} = \text{Im } BG$, since this will imply (2.10).

Let $p_1, p_2, \ldots, p_q$ be a basis for $\text{Im } B \cap \mathcal{R}$. Then there is a linearly independent set $\{u_1, u_2, \ldots, u_q\}$ such that

$$p_i = Bu_i \quad i = 1, 2, \ldots, q,$$

since if the $u_i$’s were linearly dependent then the $p_i$’s would be linearly dependent as well. If we let the input space be $\mathbb{R}^m$ it holds that $q \leq \text{dim}(\text{Im } B) \leq m$. Choose $u_{q+1}, \ldots, u_m$ such that $\{u_1, \ldots, u_m\}$ is a basis for $\mathbb{R}^m$. We want

$$BGu_i = \begin{cases} p_i & i = 1, 2, \ldots, q, \\ 0 & i = q + 1, \ldots, m \end{cases}$$

which yields $\text{Im } BG = \text{Im } B \cap \mathcal{R}$, i.e.,

$$BG[u_1, \ldots, u_m] = [p_1, \ldots, p_q, 0, \ldots, 0] = B[u_1, \ldots, u_q, 0, \ldots, 0].$$

This is achieved by

$$G := [u_1, \ldots, u_q, 0, \ldots, 0][u_1, u_2, \ldots, u_m]^{-1}.$$

We know that the reachability subspace $\mathcal{R}$ is $(A, B)$-invariant and it is obvious that $F$ in

\[(2.13) \quad \mathcal{R} = \langle A + BF | \text{Im } B \cap \mathcal{R} \rangle\]

is a friend of $\mathcal{R}$, i.e., $F \in \mathcal{F}(\mathcal{R})$.

Example 2.4. Consider

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = u_1$$

$$\dot{x}_3 = x_1 + u_2.$$ 

We will show that $V = \text{span}\{e_1, e_2\}$ is a reachability subspace.
It is easy to compute that $F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is a friend of $V$, and $\text{Im}B \cap V = \text{span}\{e_2\}$. Then, it is easy to calculate that $(A + BF)e_2 = e_1$ and $(A + BF)e_1 = e_1$. Thus $< A + BF | \text{Im}B \cap V > = V$. We note that $V_1 = \text{span}\{e_1\}$ is an $(A,B)$-invariant subspace but not a reachability subspace since $\text{Im}B \cap V_1 = 0$.

The next theorem shows that the representation (2.13) is independent of the actual choice of $F \in \mathcal{F}(\mathcal{R})$.

**Theorem 2.4.** Let $\mathcal{R}$ be a reachability subspace and let $\hat{F} \in \mathcal{F}(\mathcal{R})$, i.e. an arbitrary friend of $\mathcal{R}$. Then

$$\mathcal{R} = < A + BF | \text{Im} B \cap \mathcal{R} >.$$

**Proof**

From Theorem 2.3 follows the existence of an $F$ such that

$$\mathcal{R} = < A + BF | \text{Im} B \cap \mathcal{R} >.$$

Now let $\hat{F} \in \mathcal{F}(\mathcal{R})$ be an arbitrary friend and form

$$\hat{\mathcal{R}} = < A + B\hat{F} | \text{Im} B \cap \mathcal{R} >.$$

Since $(A + B\hat{F})\mathcal{R} \subseteq \mathcal{R}$, it holds that $\hat{\mathcal{R}} \subseteq \mathcal{R}$.

We shall show that $\mathcal{R} \subseteq \hat{\mathcal{R}}$ by induction. Clearly $\text{Im} B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}$. Assume that

$$(A + BF)^k(\text{Im} B \cap \mathcal{R}) \subseteq \hat{\mathcal{R}}.$$

Then

$$(A + BF)^{k+1}(\text{Im} B \cap \mathcal{R}) \subseteq (A + BF)\hat{\mathcal{R}} \subseteq (A + B\hat{F})\hat{\mathcal{R}} + B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}} \text{ if } B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}}.$$

If we can show the last inclusion, then it follows by induction that $\mathcal{R} \subseteq \hat{\mathcal{R}}$.

We need to introduce a lemma here in order to carry on the proof.

**Lemma 2.5.** Let $F_1 \in \mathcal{F}(\mathcal{V})$. Then $F_2 \in \mathcal{F}(\mathcal{V})$ if and only if $B(F_1 - F_2)\mathcal{V} \subseteq \mathcal{V}$.

**Proof (Proof of the lemma)**

(only if) Suppose $F_1, F_2 \in \mathcal{F}(\mathcal{V})$. Then for all $v \in \mathcal{V}$ it holds that $(A + BF_1)v \in \mathcal{V}$ and $(A + BF_2)v \in \mathcal{V}$, which implies that $B(F_1 - F_2)v \in \mathcal{V}$.

(if) Let $v \in \mathcal{V}$. Then $(A + BF_1)v + B(F_2 - F_1)v = (A + BF_2)v \in \mathcal{V}$, since the terms on the left hand side are in $\mathcal{V}$.

Now we return to the proof of the theorem. Since $\hat{\mathcal{R}} \subseteq \mathcal{R}$ it holds that

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{R} \subseteq \{\text{Lemma 2.5}\} \subseteq \mathcal{R}.$$
2.4. Maximal reachability subspaces

But $B(F - \hat{F})\hat{R} \subseteq \operatorname{Im} B$ and therefore

$$B(F - \hat{F})\hat{R} \subseteq \operatorname{Im} B \cap \mathcal{R} \subseteq \hat{R}.$$ 

Combining Theorem 2.3 and Theorem 2.4 we obtain the following result, which can be used to test whether a given subspace is a reachability subspace.

**Corollary 2.6.** Suppose $V$ is $(A, B)$-invariant and let $F \in \mathcal{F}(V)$ be an arbitrary friend of $V$. The necessary and sufficient condition for $V$ to be a reachability subspace is that

$$\langle A + BF| \operatorname{Im} B \cap V \rangle = V.$$

2.4. Maximal reachability subspaces

Consider the class $S(Z)$ of $(A, B)$-invariant subspaces contained in $Z$, and in particular reachability subspaces $\mathcal{R}$ such that $\mathcal{R} \in S(Z)$. All these satisfy

$$\mathcal{R} \subseteq S^*(Z),$$

where $S^*(Z)$ is the maximal $(A, B)$-invariant subspace in $Z$. The existence of $S^*$ is shown as follows.

**Lemma 2.7.** Let $Z$ be a subspace of $\mathbb{R}^n$. Then, the class $S(Z)$ of all $(A, B)$-invariant subspaces $S \subseteq Z$ has a maximal element $S^*(Z)$ in the sense that

$$S \subseteq S^*(Z) \text{ for all } S \in S(Z).$$

**Proof**

Note first that the set $S(Z)$ is closed under addition, i.e., if $S_1, S_2 \in S(Z)$, then $S_1 + S_2 \subseteq Z$ and

$$A(S_1 + S_2) = AS_1 + AS_2 \subseteq S_1 + S_2 + \operatorname{Im} B.$$ 

Hence, $S_1 + S_2 \in S(Z)$.

Since $Z$ is of finite dimension, there is an element $S^* \in S(Z)$ of largest dimension. If $S \in S(Z)$, then $S + S^* \in S(Z)$ and $S^* \subseteq S + S^*$. However, $S^*$ has maximal dimension and therefore, $\dim(S + S^*) = \dim S^*$, and then, $S^* = S + S^*$, that is, $S \subseteq S^*$. Thus, $S^*$ is maximal in terms of subspace inclusion.

Is there also a maximal $\mathcal{R}$ that satisfies (2.14)? Maximal in the sense that it contains all other such reachability subspaces.

**Theorem 2.8.** Let $S^*$ be the maximal $(A, B)$-invariant subspace in $Z$, and let $F \in \mathcal{F}(S^*)$. Then the maximal reachability subspace in $Z$ is

$$(2.15) \quad \mathcal{R}^* := \langle A + BF| \operatorname{Im} B \cap S^* \rangle.$$
Moreover, \( F \in \mathcal{F}(\mathcal{R}^*) \), i.e.,
\[
\mathcal{F}(\mathcal{R}^*) \supseteq \mathcal{F}(\mathcal{S}^*).
\]

We develop the proof with the help of the following two lemmas. The first of them is a refinement of Theorem 2.4, where we learn that a reachability subspace can in fact be characterized by any friend of the smaller class of friends of an \((A, B)\)-invariant subspace which generates the reachability subspace in a specific way.

**Lemma 2.9.** Let \( S \) be \((A, B)\)-invariant and let
\[
\mathcal{R} := \langle A + BF \mid \hat{B} \rangle,
\]
where \( F \in \mathcal{F}(S) \) and \( \hat{B} = \text{Im} \ B \cap S \). If \( \hat{F} \) is any matrix such that \( B(\hat{F} - F)S \subseteq S \) then we also have that
\[
\mathcal{R} = \langle A + BF \mid \hat{B} \rangle.
\]

**Remark 2.3.** Recalling Lemma 2.5 we see that the condition for \( \hat{F} \) in the above lemma amounts to \( \hat{F} \in \mathcal{F}(S) \).

**Proof**
Let \( \hat{R} := \langle A + BF \mid \hat{B} \rangle \)
and
\[
S_i := \hat{B} + (A + BF)\hat{B} + \ldots + (A + BF)^{i-1}\hat{B}.
\]
Then \( S_i \subseteq \hat{R} \).

Proceeding by induction, assume that \( S_i \subseteq \hat{R} \). Then
\[
S_{i+1} = \hat{B} + (A + BF)S_i \subseteq \hat{B} + (A + BF)S_i + B(F - \hat{F})S_i,
\]
which is included in \( \hat{R} \) if
\[
(2.16) \quad B(F - \hat{F})\hat{R} \subseteq \hat{R}.
\]
If so, \( \mathcal{R} = S_n \subseteq \hat{R} \) by induction.

We now show (2.16). Since \( \hat{F} \in \mathcal{F}(S) \) and \( \hat{B} \subseteq S \) it follows that \( \hat{R} \subseteq S \). Therefore,
\[
B(F - \hat{F})\hat{R} \subseteq B(F - \hat{F})S \subseteq \hat{B} \subseteq \hat{R}
\]
and (2.16) follows. We have thus shown that \( \mathcal{R} \subseteq \hat{R} \).

If we interchange \( F \) and \( \hat{F} \) in the calculations above, we get \( \hat{R} \subseteq \mathcal{R} \).

**Lemma 2.10.** Let \( \mathcal{R} \) and \( S \) be \((A, B)\)-invariant, and suppose that \( \mathcal{R} \subseteq S \). Then, if \( \hat{F} \in \mathcal{F}(\mathcal{R}) \), there is an \( F \in \mathcal{F}(\mathcal{R}) \cap \mathcal{F}(S) \) such that \( F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}} \).

**Proof**
Let \( \mathcal{W} \) be a subspace such that
\[
\mathcal{R} \oplus \mathcal{W} = S.
\]
Let \( \{w_1, \ldots, w_q\} \) be a basis for \( \mathcal{W} \). Since \( S \) is \((A, B)\)-invariant and \( \mathcal{W} \subseteq S \), we have
\[
Aw_i = v_i + Bu_i
\]
for some \( v_i \in S \) and \( u_i \in \mathbb{R}^m \). Now let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be such that \( Fx = \hat{F}x \) for \( x \in \mathcal{R} \) and \( Fw_i = -u_i \). Then \( F|_\mathcal{R} = \hat{F}|_\mathcal{R} \) and \( (A + BF)\mathcal{R} \subseteq \mathcal{R} \subseteq S \). Moreover, \((A + BF)w_i = v_i\), i.e., \((A + BF)\mathcal{W} \subseteq S\). Hence, \((A + BF)\mathcal{S} \subseteq S\).

We now prove Theorem 2.8.

**Proof**

We need to show that \( \mathcal{R}^* \) as defined by

\[
\langle A + BF \mid \text{Im } B \cap \mathcal{R} \rangle
\]

where \( F \) is any friend of \( \mathcal{S}^* \), is a reachability subspace in \( \mathcal{Z} \), and moreover that it is maximal.

Since \( \text{Im } B \cap \mathcal{S}^* \subseteq \mathcal{S}^* \) we have

\[
\mathcal{R}^* \subseteq \langle A + BF \mid \mathcal{S}^* \rangle = \mathcal{S}^* \subseteq \mathcal{Z}
\]

and we can always choose \( G \) such that \( \text{Im } BG = \text{Im } B \cap \mathcal{S}^* \). So \( \mathcal{R}^* \) is a reachability subspace in \( \mathcal{Z} \).

Next we show that \( \mathcal{R} \subseteq \mathcal{R}^* \) for all reachability subspaces contained in \( \mathcal{Z} \). If \( \mathcal{R} \) is an arbitrary reachability subspace in \( \mathcal{Z} \), it can be expressed as

\[
\mathcal{R} = \langle A + BF_0 \mid \text{Im } B \cap \mathcal{R} \rangle
\]

for some \( F_0 \in \mathcal{F}(\mathcal{R}) \). Clearly, \( \mathcal{R} \subseteq \mathcal{S}^* \). Moreover, by Lemma 2.10 there is an \( F_1 \in \mathcal{F}(\mathcal{S}^*) \) such that

\[
(2.17) \quad F_1|_\mathcal{R} = F_0|_\mathcal{R}.
\]

Now if \( x \in \mathcal{S}^* \) then

\[
B(F - F_1)x = (A + BF)x - (A + BF_1)x \in \mathcal{S}^*,
\]

since \( F, F_1 \in \mathcal{F}(\mathcal{S}^*) \). Hence,

\[
(2.18) \quad B(F - F_1)\mathcal{S}^* \subseteq \text{Im } B \cap \mathcal{S}^*.
\]

Consequently,

\[
(2.19) \quad \mathcal{R} = \langle A + BF_0 \mid \text{Im } B \cap \mathcal{R} \rangle
\]

\[
(2.20) = \langle A + BF_1 \mid \text{Im } B \cap \mathcal{R} \rangle
\]

\[
(2.21) \subseteq \langle A + BF_1 \mid \text{Im } B \cap \mathcal{S}^* \rangle
\]

\[
(2.22) = \langle A + BF \mid \text{Im } B \cap \mathcal{S}^* \rangle = \mathcal{R}^*,
\]

where (2.20) follows from (2.17), (2.21) follows from \( \mathcal{R} \subseteq \mathcal{S}^* \), and (2.22) follows from Lemma 2.9 and (2.18). But \( \mathcal{R} \) is arbitrary, and therefore \( \mathcal{R}^* \) is the unique maximal reachability subspace in \( \mathcal{Z} \).

We conclude this section with an example.

**Example 2.5.** Compute \( \mathcal{R}^* \) contained in \( \mathcal{Z} = \ker C \) for

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and } C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.
\]
For this purpose, we need to compute $S^*$ first. Set $S_0 = \ker C$, i.e.,

$$S_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we check if

$$AS_0 \subset S_0 + \text{Im}B.$$ 

Since

$$[S_0, B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has full dimension, clearly,

$$AS_0 \subset S_0 + \text{Im}B.$$ 

Hence,

$$S^* = \ker C = \text{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [v_1, v_2].$$

(What should we do if $S_0 \neq S^*$?) The next step is to determine a friend for $S^*$. Since $(A + BF)S^* \subseteq S^*$ implies

$$AS^* \subseteq S^* + B(-F)S^*,$$

Therefore, we form

$$A[v_1, v_2] = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 2 & 3 \end{bmatrix}$$

(2.23)

$$= \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$ 

(2.24)

The first term in (2.24) is in $S^*$ and the second term has the form $B[u_1, u_2]$. Hence, to find $F$ we must solve the system

$$-F[v_1, v_2] = [u_1, u_2],$$

i.e.,

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \text{ with solution } \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. $$

If we choose $f_{11} = f_{21} = 0$ then $A + BF = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Finally, straightforward computations yields

$$\text{Im } B \cap S^* = \text{Im} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \mathbb{R}^*.$$
2.5. Reachability under state constraints

Consider the system

\(\dot{x} = Ax + Bu\)

In this section we shall answer the following question. Let \(Z\) be an arbitrary subspace of \(\mathbb{R}^n\). Which states can be reached from the origin if we require that the trajectory lies in \(Z\)?

Before giving the main result we state some lemmas.

**Lemma 2.11.** Let \(x(t, u)\) be the solution of a controlled differential equation and let \(M\) be a subspace of \(\mathbb{R}^n\). If \(x(t, u) \in M\) for all \(t\) then \(\dot{x}(t, u) \in M\) for all \(t\).

The proof is left as an exercise for the reader.

**Lemma 2.12.** Consider the system (2.25) and let \(Z\) be a subspace of \(\mathbb{R}^n\). If \(x(t) \in Z\) for \(t \geq 0\) then \(x(t) \in S^*(Z)\) for \(t \geq 0\).

The proof is left as an exercise for the reader.

**Lemma 2.13.** Consider the system (2.25) and let \(Z\) be a subspace of \(\mathbb{R}^n\). If \(x(0) = 0\) and \(x(t) \in Z\) for \(t \geq 0\) then \(x(t) \in R^*(Z)\) for \(t \geq 0\).

**Proof**

By Lemma 2.12 we know that \(x(t) \in S^*(Z)\) for \(t \geq 0\). Now, let \(F\) be a friend of \(S^*(Z)\) and write the input as

\[ u = Fx + v. \]

Then

\[ Bv(t) = \dot{x}(t) - (A + BF)x(t) \in S^*(Z) \text{ for } t \geq 0, \]

by Lemma 2.11 and Lemma 2.12. Hence,

\[ Bv(t) \in \text{Im } B \cap S^*(Z), \]

which implies that

\[ x(t) = \int_0^t e^{(A+BF)(t-s)}Bv(s)\, ds \in (A + BF| \text{Im } B \cap S^*(Z)) = R^*(Z). \]

We can now solve the problem of reachability under constraints.

**Theorem 2.14.** Let \(S\) be the set of states that can be reached from the origin with trajectories in \(Z\), i.e.,

\(\dot{x} = Ax + Bu\)

\(S := \{ x \in Z \mid \exists t_1 : x(t_1) = x \text{ and } x(t) \in Z \forall t \in [0, t_1] \}. \)

Then,

\[ R^*(Z) = S. \]
Proof
We first show that $S \subseteq R^*(Z)$. Let $x \in S$. Then there is an input and a time $t_1$ such that $x(t) \in Z$ and $x(t_1) = x$. By Lemma 2.13 it follows that $x(t) \in R^*(Z)$ for $t \leq t_1$, and in particular, $x(t_1) \in R^*(Z)$.

Next we show that $R^*(Z) \subseteq S$. Let $F$ and $G$ be such that $R^*(Z)$ is the reachable subspace of the closed-loop system

$$\dot{x} = (A + BF)x + BGv.$$ 

Clearly, all points in $R^*(Z)$ can be reached by trajectories in $R^*(Z)$. Hence, $R^*(Z) \subseteq S$. 

$\blacksquare$