CHAPTER 4

Zeros and zero dynamics

4.1. Zero dynamics for SISO systems

Consider a linear system defined by a strictly proper scalar transfer function that does not have any common zero and pole:

$$g(s) = \alpha \frac{p(s)}{d(s)} = \alpha \frac{s^m + p_1 s^{m-1} + \cdots + p_{m-1}s + p_m}{s^n + d_1 s^{n-1} + \cdots + d_{n-1}s + d_n}.$$

The roots of the polynomial $p(s)$ are called *transmission zeros* of the system. $n-m$ is called the relative degree of the transfer function.

A minimum (controllable and observable) state space realization $(A,\ b,\ c)$ is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_n & -d_{n-1} & -d_{n-2} & \cdots & -d_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \end{bmatrix} u \\ y &= [p_m \ \cdots \ p_1 \ 1 \ 0 \cdots \ 0]x \end{aligned} \tag{4.1}$$

Now let us consider the following problem:

Problem 4.1. *Find, if possible, a control u and initial conditions x_0 such that $y(t) = 0$ $\forall t \geq 0$.*

If the problem has a solution, we can define accordingly

Definition 4.1. *The dynamics of the system* (4.1) *restricted to the set of initial conditions defined in Problem 4.1 (if the set is well defined) is called the* zero dynamics.

As we will find out, the name "zero dynamics" is due to its relation to output zeroing and its relation to transmission zeros.

Naturally, in order to keep $y = 0$, we just need to find initial conditions and a feedback control such that

$$y^{(i)} = 0, \ i = 0, 1, \ldots.$$

When we compute $y^{(i)}$, as an implication of the relative degree, we see by straightforward calculation that for (4.1)

$$cA^i b = 0, \ \text{for } i = 0, 1, \ldots, n-m-2, \ cA^{n-m-1}b \neq 0. \tag{4.2}$$

In other words, we have

$$\begin{aligned} y^{(i-1)} &= cA^{i-1}x, \ i = 1, \ldots, n-m \\ y^{(n-m)} &= cA^{n-m}x + cA^{n-m-1}bu \end{aligned}$$

REMARK 4.1. *The property (4.2) is coordinate-independent, since the relative degree is obviously so. We leave the proof as an exercise.*

The implication (4.2) leads to the fact that the rows

$$cA^{i-1}, \ i = 1, \ldots, n-m$$

are linearly independent. This can easily be shown by contradiction. We leave this as an exercise.

We now do a coordinate change by letting

$$\begin{aligned} \xi_i &:= cA^{i-1}x \qquad i = 1, \ldots, n-m \\ z_i &:= x_i \qquad i = 1, \ldots, m, \end{aligned} \tag{4.3}$$

where one can easily verify that the $z_i's$ are linearly independent from the $\xi_i's$. Then the system can be written as

$$\begin{aligned} \dot{z} &= Nz + P\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{n-m-1} &= \xi_{n-m} \\ \dot{\xi}_{n-m} &= Rz + S\xi + \alpha u \\ y &= \xi_1 \end{aligned} \tag{4.4}$$

where

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -p_m & -p_{m-1} & \cdots & \cdots & -p_1 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4.5}$$

and

$$\xi = (\xi_1, \cdots, \xi_{n-m})^T.$$

(4.4) is called *the normal form* of (4.1).

In order to keep $y(t) = 0$, we must have $\xi_1 = \xi_2 = \cdots = \xi_{n-m} = 0$ and

$$u = \frac{1}{\alpha}(-Rz - S\xi).$$

So the zero dynamics is defined on the subspace

$$Z^* := \{x : \ cA^i x = 0, \ i = 0, \ldots, n-m-1\}$$

and represented by

$$\dot{z} = Nz.$$

The eigenvalues of N are zeros of $g(s)$.

Question: What is $\mathcal{V}^*$ for (4.1)?

4.2. Zero dynamics of MIMO systems

In this section we discuss zeros and zero dynamics of a multivariable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{4.6}$$

where (A, B, C) is minimal and A is $n \times n$. From now we always assume that both B and C have full rank. For the time being we assume that the number of inputs and the number of the outputs are the same (a square system), namely B is $n \times m$ and C is $m \times n$. Correspondingly we also have the frequency domain representation of (4.6) as

$$G(s) = C(sI - A)^{-1}B.$$

Unfortunately we do not have a straightforward way to extend the concept of transmission zero for a SISO system to the MIMO case. So instead we first study the *output zeroing problem*:

PROBLEM 4.2. *Find, if possible, a control u and initial conditions x_0 such that $y(t) = 0$ $\forall t \geq 0$.*

Similarly we define:

DEFINITION 4.2. *The dynamics of the system* (4.6) *restricted to the set of initial conditions defined in Problem 4.2 (if the set is well defined) is called the* zero dynamics*.*

In the SISO case, we used the normal form to solve the problem, where the relative degree played an important role. Thus we first generalize this concept to the MIMO case.

DEFINITION 4.3. *System* (4.6) *is said to have* relative degree $(r_1, \ldots, r_m)$ *if for $i = 1, \ldots, m$*

$$\begin{aligned} c_i A^j B &= 0, \ \forall j = 0, \ldots, r_i - 2 \\ c_i A^{r_i - 1} B &\neq 0, \end{aligned}$$

and the matrix

$$L := \begin{bmatrix} c_1 A^{r_1 - 1} B \\ \vdots \\ c_m A^{r_m - 1} B \end{bmatrix} \tag{4.7}$$

is nonsingular.

Similar to the SISO case, we do a coordinate change:

$$\xi_j^i = c_i A^{j-1} x, \ i = 1, \ldots, m \text{ and } j = 1, \ldots, r_i,$$

and complete the coordinates by choosing

$$z_i = p_i x, \text{ such that } p_i B = 0, \ i = 1, \ldots, n - (r_1 + \cdots + r_m).$$

Question: why can we find such $p_i's$ that $p_i B = 0$?

Now we can also derive a normal form as

$$
\begin{array}{rcl}
\dot{z} & = & Nz + P\xi \\
\dot{\xi}_1^i & = & \xi_2^i \\
 & \vdots & \\
\dot{\xi}_{r_i-1}^i & = & \xi_{r_i}^i \\
\dot{\xi}_{r_i}^i & = & R_i z + S_i \xi + c_i A^{r_i-1} B u \\
y_i & = & \xi_1^i, \ i = 1, \ldots, m
\end{array}
\tag{4.8}
$$

where

$$\xi := (\xi_1^1, \cdots, \xi_{r_1}^1, \cdots, \xi_1^m, \cdots, \xi_{r_m}^m)^T.$$

In order to keep $y(t) = 0$, we must have $\xi = 0$ and $u = L^{-1}(-Rz - S\xi)$, where

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}, \ S = \begin{bmatrix} S_1 \\ \vdots \\ S_m \end{bmatrix}. \tag{4.9}$$

Theorem 4.1. *If system* (4.6) *has relative degree* $(r_1, \ldots, r_m)$, *then the zero dynamics is defined on the subspace*

$$Z^* := \{x : \ c_i A^{j-1} x = 0, \ i = 1, \ldots, m \text{ and } j = 1, \ldots, r_i\}.$$

The zero dynamics under the normal form (4.8) *is represented by*

$$\dot{z} = Nz.$$

Definition 4.4. *The eigenvalues of* N *are called the* transmission zeros *of system* (4.6).

We can define a feedback transformation

$$u = Fx + v = L^{-1}(-Rz - S\xi) + v.$$

A straight calculation shows $\mathcal{V}^*$ in this case coincides with the output zeroing subspace:

$$\xi = 0.$$

Now the question is how we interpret the transmission zeros in the frequency domain.

Proposition 4.2. *Let*

$$P_\Sigma(s) := \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix}. \tag{4.10}$$

A complex number s_0 *is a transmission zero of the system* (4.6) *if and only if*

$$\text{rank } P_\Sigma(s) < n + m.$$

Remark 4.2. *The matrix in* (4.10) *is called* the system (Rosenbrock) matrix *of* (A, B, C).

PROOF
The normal form (4.8) is obtained from (4.6) via a coordinate transformation $x = Q\hat{x}$ where $\hat{x} = (z^T, \xi^T)^T$ and Q is defined accordingly. In the normal form, where $\hat{A} = Q^{-1}AQ,\ \hat{F} = FQ,\ \hat{B} = Q^{-1}B,\ \hat{C} = CQ$, we have

$$\hat{A} + \hat{B}\hat{F} = \begin{bmatrix} N & P \\ 0 & H \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix} 0 & C_1 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}$$

where H and C_1 and B_1 are determined by the normal form and the triple $(H,\ B_1,\ C_1)$ is both controllable and observable. Because of the special forms of $\hat{B}$ and $\hat{C}$, it is not so difficult to see that

$$\text{rank} \begin{bmatrix} s_0 I - \hat{A} - \hat{B}\hat{F} & \hat{B} \\ -\hat{C} & 0 \end{bmatrix} < n + m$$

if and only if s_0 is an eigenvalue of N, namely a transmission zero. Then the proposition is proven since

$$\begin{bmatrix} s_0 I - \hat{A} - \hat{B}\hat{F} & \hat{B} \\ -\hat{C} & 0 \end{bmatrix} = \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}.$$

∎

With the above proposition, we can easily define transmission zeros even for non-square systems.

Now suppose system (4.6) has m inputs and p outputs, then

DEFINITION 4.5. *A complex number s_0 is called a* transmission zero *of the system* (4.6) *if*

$$\text{rank} \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} < n + \min(m, p). \tag{4.11}$$

REMARK 4.3. *In the multivariable case a complex number can be both a zero and a pole (eigenvalue of A). If (A, B, C) is not minimal, the above definition may include so-called input/output decoupling zeros. In general, $n + \min(m, p)$ should be replaced by the highest possible rank of the system matrix.*

Now we give an alternative characterization of transmission zeros, which was used in some literature. Especially, this characterization will help us understand some of the historic results on the output regulation problem, which will be studied in Chapter 7. For this purpose, we need to introduce the following lemma.

LEMMA 4.3. [**16**] *For any $m \times n$ polynomial matrix $P(s)$ there exists a sequence of elementary (row and column) operations that transforms $P(s)$ to*

$$\hat{P}(s) = diag_{m\times n}(\Psi_1(s), \Psi_2(s), \ldots, \Psi_r(s), 0, \ldots, 0),$$

where $\Psi_1, \Psi_2, \ldots$ *are monic non-zero polynomials satisfying* $\Psi_1 \mid \Psi_2 \mid \Psi_3 \cdots$, *namely each* Ψ_i *divides the next one.*

These non-zero polynomials are called the (non-trivial) invariant factors *of* $P(s)$.

Definition 4.6. *The non-trivial invariant factors of* $P(s)$ *are called the* transmission polynomials *of the system. The product of the transmission polynomials is called the* zero polynomial.

Lemma 4.4. s_0 *is a transmission zero if and only if it is a root of the zero polynomial.*

The following result clarifies why “transmission” was in the name.

Lemma 4.5. *Suppose* s_0 *is a transmission zero, namely there exist* x_0 *and* u_0 *such that*

$$P_\Sigma(s_0)\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0.$$

Let $u(t) = -e^{s_0 t}u_0$. *Then the output resulting from the initial state* x_0 *and input* u *is zero, just as in the square case.*

The proof is left as an exercise to the reader.

Theorem 4.6. *Suppose* s_0 *is not a pole. Then* s_0 *is a transmission zero of* (4.6) *if and only if* rank $W(s_0) < \min(m,p)$, *where* $W(s) = C(sI-A)^{-1}B$.

Proof
Consider

$$\begin{bmatrix} I & 0 \\ C(sI-A)^{-1} & I \end{bmatrix}\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} sI-A & B \\ 0 & W(s) \end{bmatrix}. \tag{4.12}$$

Since the first matrix in (4.12) is nonsingular at all s that are not poles, the Rosenbrock matrix drops rank at s_0 if and only if $W(s)$ drops rank at s_0. ∎

4.3. Zeros and system inversion

We now summarize the discussions in the previous section.

Theorem 4.7. *Consider system* (4.6) *and suppose that dim*(Im B) $= m$. *Then the transmission zeros of the system are the eigenvalues of* $(A+BF)|_{\mathcal{V}^*}$, *namely, those eigenvalues of* $A+BF$ *that are invariant over the class* $\mathcal{F}(\mathcal{V}^*)$ *of friends* F *of* $\mathcal{V}^*$. *In particular,* $\dim \mathcal{V}^*/\mathcal{R}^*$ *is equal to the number of transmission zeros.*

In the case where the system is square and has a relative degree $(r_1, \ldots, r_m)$, we see that

$$\mathcal{V}^* \cap \operatorname{Im} B = 0. \tag{4.13}$$

Thus $\mathcal{R} = 0$.

In general (including the non-square case), condition (4.13) together with the condition that B has full column rank are the conditions for the system being **invertible**, namely, from a given initial point x_0, any two controls u_1 and u_2 that produce the same output are necessarily equal. These are also the conditions that guarantee the uniqueness of the zero dynamics.

PROOF
Partition the state space as in (3.15), and consider

$$\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} = \begin{bmatrix} sI-A-BF & B \\ -C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} sI-P_{11}-B_1F_{11} & -P_{12}-B_1F_{12} & -P_{13}-B_1F_{13} & B_1 & 0 \\ 0 & sI-P_{22} & -P_{23} & 0 & 0 \\ 0 & 0 & sI-P_{33}-B_3F_{23} & 0 & B_3 \\ 0 & 0 & -C_3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} sI-P_{11}-B_1F_{11} & B_1 & -P_{12}-B_1F_{12} & -P_{13}-B_1F_{13} & 0 \\ 0 & 0 & sI-P_{22} & -P_{23} & 0 \\ 0 & 0 & 0 & sI-P_{33}-B_3F_{23} & B_3 \\ 0 & 0 & 0 & -C_3 & 0 \end{bmatrix}.$$

Since $\mathcal{R}^*$ is a reachability subspace, $(P_{11}+B_1F_{11}, B_1)$ is reachable and $\begin{bmatrix} sI-P_{11}-B_1F_{11} & B_1 \end{bmatrix}$ has full rank for all $s \in \mathbb{C}$.

If we can show that

$$\begin{bmatrix} sI-P_{33}-B_3F_{23} & B_3 \\ -C_3 & 0 \end{bmatrix} \text{ has full rank for all } s \in \mathbb{C}, \tag{4.14}$$

then the Rosenbrock matrix drops rank for exactly those s_0 for which $\big(sI-P_{22}\big)$ drops rank, i.e,

$$\{\text{zeros}\} = \{\text{eigenvalues of } P_{22}\}.$$

Thus, it remains to show the claim in (4.14). Since

$$\begin{bmatrix} sI-P_{33}-B_3F_{23} & B_3 \\ -C_3 & 0 \end{bmatrix} = \begin{bmatrix} sI-P_{33} & B_3 \\ -C_3 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -F_{23} & I \end{bmatrix}$$

it is enough to show that

$$Q(s) := \begin{bmatrix} sI-P_{33} & B_3 \\ -C_3 & 0 \end{bmatrix}$$

has full rank for all $s \in \mathbb{C}$. To this end, suppose the contrary. Then for some $s_0 \in C$ there is a nonzero vector $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$ such that $Q(s_0)\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$, i.e.,

$$\begin{cases} P_{33}x_0 &= s_0x_0 - B_3u_0 \\ C_3x_0 &= 0. \end{cases}$$

The vector x_0 is nonzero, since $x_0 = 0$ would imply $B_3u_0 = 0$ and contradict the assumption that

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix}$$

has linearly independent columns. Define the subspace

$$\mathcal{K} := \operatorname{Im} \begin{bmatrix} 0 \\ 0 \\ x_0 \end{bmatrix} \subseteq \mathbb{R}^n.$$

Since $C_3 x_0 = 0$, it follows that $\mathcal{K} \subseteq \ker C$. Moreover, $\mathcal{V}^* \oplus \mathcal{K}$ is (A, B)-invariant. To see this, form

$$A \begin{bmatrix} 0 \\ 0 \\ x_0 \end{bmatrix} = \begin{bmatrix} P_{13}x_0 \\ P_{23}x_0 \\ P_{33}x_0 \end{bmatrix} = \begin{bmatrix} P_{13}x_0 \\ P_{23}x_0 \\ 0 \end{bmatrix} + s_0 \begin{bmatrix} 0 \\ 0 \\ x_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix} u_0. \tag{4.15}$$

The three terms in the right hand side of (4.15) belong to $\mathcal{V}^*$, $\mathcal{K}$ and Im B, respectively. Hence, $\mathcal{V}^* \oplus \mathcal{K}$ is an (A, B)-invariant subspace in $\ker C$, which contradicts the maximality of $\mathcal{V}^*$. ∎

4.4. An illustration of zero dynamics: high gain control[1]

Zeros are system invariants, in the sense that coordinate transformations or feedback transformations do not alter their location. In some cases this can have a detrimental effect on the behavior of a system, if for example the input-output gain of the system drops substantially for frequencies associated with a zero while we request "good controllability" near those same frequencies. Zeros in the right half plane are especially nasty. Systems with this type of zeros are called *non-minimum phase*, because, in the scalar IO case, there exist systems with the same gain-frequency plot but smaller phase. When a system has non-minimum phase zeros, high-gain feedback (e.g. matrix F with large norm) cannot be used to stabilize the system, since it can be shown that some of the poles of the feedback system tend to its zeros as $\|F\|$ increases. This is a similar phenomenon as that what occurs with root-locus diagrams. Consequently, the selection of feedback matrices in the non-minimum phase case is very delicate, since feedback should be high enough to have some effect but simultaneously low enough to avoid instabilities. Finally, it can be shown that the sensitivity of a non-minimum phase system to disturbances acting at the input of the plant is severely limited both when the open-loop plant has unstable poles and non-minimum phase zeros. More on this can be read in [**5**].

In the rest of the section, we use the geometric interpretation of zeros, the zero dynamics, to explain the effects of high gain controls. For the sake of simplicity, we consider a SISO system:

$$g(s) = \alpha \frac{p(s)}{d(s)} = \alpha \frac{s^m + p_1 s^{m-1} + \cdots + p_{m-1}s + p_m}{s^n + d_1 s^{n-1} + \cdots + d_{n-1}s + d_n}. \tag{4.16}$$

[1] The topics discussed in the section are more advanced

It is well known that if the system is minimum phase, then the following output feedback control

$$u = -k^{n-m} q(\frac{s}{k}) y$$

stabilizes the system when $k > 0$ is sufficiently large, where $q(s) = q_0 + \cdots + q_{n-m-1} s^{n-m-1}$ is any $(n-m-1)$-order polynomial such that $s^{n-m} + q(s)$ is Hurwitz.

It is easy to see that in the normal form, the closed-loop system becomes

$$\begin{array}{rcl} \dot{z} &=& Nz + P\xi_1 \\ \dot{\xi}_1 &=& \xi_2 \\ &\vdots& \\ \dot{\xi}_{n-m-1} &=& \xi_{n-m} \\ \dot{\xi}_{n-m} &=& Rz + S\xi - \sum_{i=0}^{n-m-1} q_i k^{n-m-i} \xi_{i+1} \\ y &=& \xi_1, \end{array} \tag{4.17}$$

where N and P are defined in (4.5). If we let $\epsilon := \frac{1}{k}$ and $\tilde{\xi}_i := \epsilon^{i-1}\xi_i$, then we have

$$\begin{array}{rcl} \dot{z} &=& Nz + P\tilde{\xi}_1 \\ \epsilon\dot{\tilde{\xi}}_1 &=& \tilde{\xi}_2 \\ &\vdots& \\ \epsilon\dot{\tilde{\xi}}_{n-m-1} &=& \tilde{\xi}_{n-m} \\ \epsilon\dot{\tilde{\xi}}_{n-m} &=& \epsilon R(\epsilon) z + \epsilon S(\epsilon)\tilde{\xi} - \sum_{i=0}^{n-m-1} q_i \tilde{\xi}_{i+1}. \end{array} \tag{4.18}$$

Then by singular perturbation arguments, we have $\tilde{\xi} = O(\epsilon) z$, as $t \to \infty$ and k is sufficiently large, where $O(0) = 0$. In the first equation, we then have

$$\dot{z} = Nz + PO_1(\epsilon) z.$$

If N is stable (minimum phase condition), then the above subsystem is also stable if ϵ is small enough!

Another example is the so-called cheap control problem . Here the idea is that the cost of control is “cheap”. For a minimum realization of (4.16)

$$\begin{array}{rcl} \dot{x} &=& Ax + bu \\ y &=& cx, \end{array} \tag{4.19}$$

the cheap control problem is characterized by the presence of ϵ in the cost functional

$$J = \frac{1}{2} \int_0^\infty (\|cx\|^2 + \epsilon^2 \|u\|^2) dt \tag{4.20}$$

where ϵ is a small positive constant.

As is well known, the optimal control of (4.20) is a linear feedback control $u = F_\epsilon x$ where

$$F_\epsilon = -\frac{1}{\epsilon^2} B^T P_\epsilon$$

and P_ϵ is the positive semidefinite solution of the Riccati equation

$$A^T P_\epsilon + P_\epsilon A + c^T c = \frac{1}{\epsilon^2} P_\epsilon b b^T P_\epsilon. \tag{4.21}$$

By the classical results of linear optimal control , we know the closed-loop system is stable. Now an interesting question is, what happens to the optimal trajectories if we let $\epsilon \to 0$?

It turns out that in order to have the trajectories stay bounded, it is necessary and sufficient to assume that (4.16) has relative degree 1 ($m = n-1$) and no transmission zeros on the imaginary axis. It is not so difficult to see [**10**] (although we will not derive it here) that under these conditions as $\epsilon \to 0$ the optimal trajectories tend to the solutions of the following optimal control problem:

$$\min \frac{1}{2} \int_0^\infty \|v\|^2 dt$$

subject to $\dot{z} = Nz + Pv$ being asymptotically stable.

When N does not have any eigenvalue on the imaginary axis, the optimal control problem is well posed.

We can show that the optimal solution will lead to the zero dynamics equation if the original system is minimum phase; otherwise it will lead to the following system

$$\dot{z} = N_1 z,$$

where the eigenvalues of N_1 are either stable transmission zeros, or the mirror images of unstable transmission zeros.