KTH Matematik

SF2842: Geometric Control Theory

Solution to Homework 1

For reference only

You may discuss the problems in group (maximal **two** students in a group), but each of you **must** write and submit your own report. Write the name of the person you cooperated with.

---

1. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= Cx = (1\ 0\ 0\ 0)x. \end{aligned}$$

(a) Is the system observable? .... (1p)

(b) Compute $\mathcal{V}^*$ and $\mathcal{R}^*$ contained in $\mathcal{V}^*$, and find (parameterize) ALL friends F of $\mathcal{V}^*$. .... (3p)

(c) Let F be any friend of V^*, $A_F = A+BF$, and $\Omega_F = (C^T, A_F^T C^T, \cdots, (A_F^3)^T C^T)^T$. What is the dimension of $ker\ \Omega_F$? .... (1p)

Solution:

a. $y = x_1 = 0 \ \Rightarrow\ x_3 = 0 \ \Rightarrow\ x_2 = 0 \ \Rightarrow\ x_4 = 0$. Yes.

b. $y = x_1 = 0 \ \Rightarrow\ \dot{x}_1 = x_3 = 0 \ \Rightarrow\ x_2 + x_3 + u_2 = 0 \ \Rightarrow\ u_2 = -x_2 - x_3 \ \Rightarrow\ \underline{\mathcal{V}^* = \text{span}\{e_2, e_4\}}$

$$\Rightarrow \begin{array}{ll} \dot{x}_2 & = -x_2 + x_4 + u_1 \\ \dot{x}_4 & = -x_4 \end{array} \Rightarrow \underline{\mathcal{R}^* = \text{span}\{e_2\}}$$

c. $\underline{\dim \ker \Omega_F = \dim \mathcal{V}^* = 2}$.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= (0\ 1\ 1\ 0)x, \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4)^T$.

(a) Given $x(0) = (0,0,0,0)^T$, find (parameterize) all points $\bar{x}$ in $ker\ C$ that can be reached from $x(0)$ in **any** given finite time $T > 0$ with some control (i.e. $x(T) = \bar{x}$), while the trajectory of $x(t)$ is a straight line. (Here, $x(t)$ is the solution of $\dot{x} = Ax + Bu$, $x(0) = 0$.) . . . . (3p)

(b) If we let $x(0) = (0,1,-1,0)^T$, and $\bar{x} = (1,k,-k,0)^T$, for what values of k $\bar{x}$ can be reached in **some** finite time from $x(0)$ while $x_4(t) = 0\ \forall t \geq 0$? . . . . (2p)

Solution:

a. $\ker C = \text{span}\{e_1, e_4, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\}$.

Since $\text{Im}B = \text{span}\{e_1, e_4\}$, we only need to verify if $\text{span}\{\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix}\}$ is (A,B)-invariant:

$$A \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ 0 \\ \beta \\ \alpha - 2\beta \end{pmatrix} \in \text{span}\{e_1, e_4\} \cup \text{span}\{\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix}\}$$

$\Rightarrow \beta = 0 \ \ \Rightarrow\ \text{span}\{e_1\}$ is (A,B)-invariant.
Setting $x_2 = x_3 = x_4 = 0 \ \Rightarrow\ u_2 = -x_1 - x_2 + 2x_4$

$$\Rightarrow\ (A+BF) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 9 \end{pmatrix},\ \langle A+BF | \text{span}\{e_1\}\rangle = \text{span}\{e_1\}.$$

$\Rightarrow$ $\underline{\text{span}\{e_1\}\text{ is the controllability subspace (the only dim1 one).}}$

b. Setting $x_4 \equiv 0 \ \Rightarrow\ u_2 = -x_1 - x_2 + 2x_4$

$$\Rightarrow \begin{array}{ll} \dot{x}_1 & = -x_1 + u_1 \\ \dot{x}_2 & = x_3 \\ \dot{x}_3 & = -x_3 \end{array} \ \Rightarrow\ \begin{array}{ll} x_3(t) & = e^{-t}x_3(0) = -e^{-t} \\ x_2(t) & = x_2(0) - \int_0^t e^{-s}ds = x_2(0) + (e^{-t}-1) = e^{-t} \end{array}$$

$\Rightarrow\ \underline{k < 1}$

3. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C = (2\ 3\ 1).$$

(a) Find the minimum constraint on E such that DDP is solvable. . . . . (2p)

(b) Find a $u = Fx$ that solves the DDP problem while makes the closed-loop system stable, i.e. $A + BF$ has only eigenvalues with negative real part. . . (2p)

(c) Verify that there exists an $E \in V^*$ such that (A, E) is controllable. Explain why even in this case the DDP problem is solvable (namely $w(t)$ will not at all influence the output)..(1p)

Solution:

a. $y = 2x_1 + 3x_2 + x_3 \;\Rightarrow\; \dot{y} = 2\dot{x}_1 + 3\dot{x}_2 + \dot{x}_3 = 2x_2 + 3x_3 + x_1 + u$
$\Rightarrow\; u = -x_1 - 2x_2 - 3x_3 + k(2x_1 + 3x_2 + x_3) \;\Rightarrow\; \underline{V^* = \ker C}$

b. Let $p_d(s) = (s^2 + 3s + 2)(s + 1) = s^3 + 4s^2 + 5s + 2$, i.e., $\dot{x}_3 = -2x_1 - 5x_2 - 4x_3$
$\Rightarrow\; u = -2x_1 - 5x_2 - 4x_3 - x_1 = \underline{-3x_1 - 5x_2 - 4x_3}$
$\Rightarrow\; k = -1 \;\Rightarrow\; A + BF$ is a stable matrix

c. $E = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $AE = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $A^2E = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ $\det \begin{pmatrix} 1 & 0 & -2 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix} = -1 + 2(-4) \neq 0.$

4. Consider

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_3 + u_1 \\
\dot{x}_2 &= x_2 + x_3 - u_1 \\
\dot{x}_3 &= -x_3 + 2x_4 + u_2 \\
\dot{x}_4 &= -x_1 - x_2 + x_4 + u_1 \\
y_1 &= x_1 + x_2 \\
y_2 &= x_4
\end{aligned}$$

(a) What is the relative degree for the system? .... (1p)

(b) Convert the system into the normal form and compute the zero dynamics.(3p)

(c) When $y(t) = 0\ \forall t \geq 0$, does it always imply $\lim_{t\to\infty} x(t) = 0$? .... (1p)

Solution:

a. $y_1 = x_1 + x_2 = 0 \;\Rightarrow\; \dot{x}_1 + \dot{x}_2 = x_1 + x_2 + 2x_3 = 0 \;\Rightarrow\; x_3 = 0 \;\Rightarrow\; u_2 = -2x_4$
$y_2 = x_4 \;\Rightarrow\; \dot{x}_4 = x_4 + u_1 = 0 \;\Rightarrow\; u_1 = -x_4$.
$\Rightarrow$ $\underline{\text{relative degree} = (2, 1)}$

b. $\xi_1^1 = y_1 = x_1 + x_2$,
$\xi_2^1 = x_1 + x_2 + 2x_3$,
$\xi_1^2 = y_2 = x_4$,
Let $z = x_2 + x_4$

$$\Rightarrow \begin{cases} \dot{\xi}_1^1 &= \xi_2^1 \\ \dot{\xi}_2^1 &= \xi_1^1 + 4\xi_1^2 + u_2 \\ \dot{\xi}_1^2 &= - - \xi_1^1 + \xi_1^2 + u_1 \\ \dot{z} &= z - \frac{3}{2}\xi_1^1 + \frac{1}{2}\xi_2^1 \\ y_1 &= \xi_1^1 \\ y_2 &= \xi_1^2 \end{cases}$$

zero dynamics: $\underline{\dot{z} = z}$

c. No, since $\dot{z} = z$ is unstable.