KTH Matematik

SF2842: Geometric Control Theory

Homework 1

Due February 10, 16:59, 2015

You may discuss the problems in group (maximal **two** students in a group), but each of you **must** write and submit your own report. Write the name of the person you cooperated with.

---

1. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= Cx = (1\ 0\ 0\ 0)x. \end{aligned} \tag{1}$$

(a) Is the system controllable? . . . (1p)

(b) Let $x(t, u)$ denote the solution to (1) with control $u(t)$ and initial condition $x(0, u) = x_0$. Compute the subspace S of initial conditions x_0 that make $x(t, u) \in Ker\ C\ \forall t \geq 0$ for some $u(t)$, and design such a $u(t)$ as feedback control. . . . (3p)

(c) For any $x_0 \in S$, where S is the subspace you computed in (b), and any $t_1 > 0$, can we always find a $u(t)$ such that $x(t_1, u) = 0$? . . . (1p)

(d) For any $x_0 \in S$, where S is the subspace you computed in (b), and any $t_1 > 0$, can we always find a $u(t)$ such that $x(t_1, u) = 0$, and $x(t, u) \in S,\ 0 \leq t \leq t_1$? (2p)

Answer:

(a). The system is controllable since the matrix $\begin{pmatrix} B & AB \end{pmatrix}$ already has rank 4.

(b) $S = V^* = span\{e_3, e_4\}$.

(c) Yes, since the system is controllable.

(d) Yes, since S is a reachability subspace.

2. Consider an observable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned}$$

where $x \in R^n,\ u \in R^1,\ y \in R^1$.

(a) Show the controllable subspace $\mathcal{R}$ is $(A + BF)$-invariant for any F. . . . (1p)

(b) List all controllability subspaces. .... (1p)

(c) Show that $(C,\ A+BF)$ is also observable for almost all F, namely the elements of those F that make $(C,\ A+BF)$ not observable can be defined by a set of algebraic constraints. .... (3p)

Answer:

(a) By the definition of the reachable subspace, $\mathcal{R} = \langle A|\text{Im } B\rangle$ is an A-invariant subspace that contains, at least, Im B. For any $x \in \mathcal{R}$, and any F, we have that $(A+BF)x = Ax+BFx$. $\mathcal{R}$ is A-invariant implies that $Ax \in \mathcal{R}$. Together with the fact that $BFx \in \text{Im } B$ and the definition of a subspace, we know that $(A+BF)x \in \mathcal{R}$. Thus, $\mathcal{R}$ is $(A+BF)$-invariant for any F.

(b) By definition, a reachability subspace is $\langle A+BF|\text{Im } BG\rangle$ for some F and G. Note that for a SISO system, the G will be a scalar. If $G \neq 0$, then subspace Im BG is equal to Im B, and the reachability subspace becomes $\langle A+BF|\text{Im } B\rangle = \langle A|\text{Im } B\rangle$ – the reachable subspace. If $G = 0$, then we have a trivial reachability subspace $\{0\}$. These two are the only possible reachability subspace for a SISO system.

(c) In the Hautus test of the pair $(C, A+BF)$, if it is unobservable, then there is an s such that the matrix $\begin{pmatrix} sI - A - BF \\ C \end{pmatrix}$ dose not have full column rank. However, we just need to check for the transmission zeros here to find s, since if s is not a transmission zero, the Rosenbrock matrix $\begin{pmatrix} sI - A & B \\ -C & 0 \end{pmatrix}$ will have full column rank, and $\begin{pmatrix} sI - A - BF \\ C \end{pmatrix}$ will also have full column rank. This implies that a necessary condition for $(C, A+BF)$ to be unobservable pair is that for a transmission zero s_0 of the system, the matrix $s_0I - A - BF$ is singular, i.e. $\rho(s_0) = det\ (s_0I - A - BF) = 0$. $\rho(s_0)$ defines a polynomial of the elements of F. Since the number of transmission zero is strictly less than n, the set that is defined by the necessary condition is of measure zero.

3. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & a & 0 \\ 2 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C = (1\ 0\ 0),\ E = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

where a and d_1, d_2, d_3 are constants.

(a) For what E is DDP solvable? .... (2p)

(b) For what a can we find a $u = Fx$ that solves the DDP problem while makes the closed-loop system stable? i.e. $A + BF$ has only eigenvalues with negative real part. .... (2p)

(c) What is R^*? .... (1p)

Answer:

(a) $V^* = span\{e_2\}$, so DDP is solvable iff $d_1 = d_3 = 0$.

(b) Only when $a < 0$ does the system have a stable zero dynamics.

(c) $\mathcal{R}^* = \{0\}$.

4. Consider

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_3 + u_1 \\
\dot{x}_2 &= -x_1 + x_3 - u_1 \\
\dot{x}_3 &= x_2 - x_3 + x_4 + u_2 \\
\dot{x}_4 &= 2x_1 + x_4 + u_1 \\
y_1 &= x_1 + x_2 \\
y_2 &= x_4
\end{aligned}$$

(a) What is the relative degree for the system? .... (1p)

(b) Convert the system into the normal form and compute the zero dynamics.(3p)

(c) When $y(t) = 0\ \forall t \geq 0$, what happens to $x(t)$ as $t \to \infty$? .... (1p)

Answer:

(a) The system has a relative degree $(2, 1)$.

(b) $\xi_1^1 = x_1 + x_2$, $\xi_1^2 = x_3$, $\xi_2^1 = x_4$. One can pick, for instance, $z = x_2 + x_4$ and get that the zero dynamics is $\dot{z} = -z$.

(c) $x(t) \to 0$ as $t \to \infty$ since the zero dynamics is stable.