KTH Matematik

SF2842: Geometric Control Theory

Solution to Homework 3

Due March 11, 16:50pm, 2015

You may discuss the problems in group (maximal two students in a group), but each of you **must** write and submit your own report. Write the name of the person you cooperated with.

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1. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= u,\end{aligned}$$

a. Show that the system is locally strongly accessible. [2p]

b. Show that the system is not controllable. (Hint: try to find some points that are not reachable from a given point) [2p]

Answer:

a. The strong accessibility distribution $R_c(x) = span\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\} = R^2$ for any x. So the system is locally strongly accessible for any x.

b. Since the derivative $\dot{x}_1 \geq 0$, it is impossible to decrease x_1 by using any control input. Hence, the system is not controllable.

2. Determine and justify if each of the following statements is true or false.

- Consider the consensus control problem in R^2 with N agents: $\dot{x}_i = Ax_i + Bu_i,\ x_i \in R^2,\ u_i \in R,\ i = 1, \cdots, N$, (A, B) is controllable and the neighbor graph is connected. If the initial positions of the agents are contained in a disc, then as a consensus control (i.e. a control that assures consensus reaching as $t \to \infty$) $u_i = K\sum_{j \in N_i}(x_j - x_i)$ is applied, no agent can move outside the disc at any time. [2p]
- Consider a smooth nonlinear control system defined in a neighborhood N of the origin

$$\dot{x} = f(x) + g(x)u,$$

where $f(0) = 0$, and $x = 0$ of $\dot{x} = f(x)$ is unstable. If the system is not controllable, then $x = 0$ is not asymptotically stabilizable by any feedback control. [2p]

Answer:

a. False. The average state $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is governed by the system $\dot{\bar{x}} = A\bar{x}$, and is not guaranteed to stay in the original disc (even for stable A matrix).

b. False. The statement is false even for linear systems. Consider the case that the uncontrollable modes are stable while the controllable modes has unstable eigenvalues.

3. Consider

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + x_2 + x_1^3 + 2x_1^2 x_2\\ \dot{x}_2 &= -x_1^3 - x_2,\end{aligned}$$

where α is constant.

Determine the stability of the origin by dividing α into different ranges so that you use respectively

a. The principle of stability in first approximation to analyze the stability; [1p]

b. The center manifold theory to analyze the stability. [2p]

Answer:

a. When $\alpha < 0$ the system exponentially stable. When $\alpha > 0$, the system is unstable. When $\alpha = 0$, nonlinear stability analysis is needed.

b. By introducing the new variable $z = x_1 + x_2$ and $y = x_2$, the center manifold is approximated by $y = -z^3 + \mathcal{O}(z^5)$. On the center manifold, the system behaviors as $\dot{w} = -2w^5 + \mathcal{O}(w^7)$ and is asymptotically stable. Hence, the original system is also asymptotically stable when $\alpha = 0$.

4. Consider in a neighborhood N of the origin

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_2 + u\\ \dot{x}_2 &= x_1 - x_2^3 + x_3 - u\\ \dot{x}_3 &= 2x_2^3 - 2x_3 + 2u\\ y &= 2x_2 + x_3.\end{aligned}$$

a. Convert the system locally into the normal form. [2p]

b. What is the zero dynamics? [1p]

c. Is the system exactly linearizable (without considering the output) around the origin? [2p]

Answer:

a. The relative degree is 2, so $\xi_1 = h(x) = 2x_2 + x_3$ and $\xi_2 = L_f h(x) = 2x_1$. One can find, for instance, $z = x_1 + x_2$ so that $\frac{\partial z}{\partial x} \cdot g(x) = 0$. The normal form will then become:

$$
\begin{aligned}
\dot{z} &= -(z - \frac{\xi_2}{2}) - (z - \frac{\xi_2}{2})^3 + \xi_1 + \frac{\xi_2}{2} + (\frac{\xi_2}{2})^3 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= 2z - \xi_2 + 2(\frac{\xi_2}{2})^3 + 2u \\
y &= \xi_1
\end{aligned}
$$

b. The zero dynamics is $\dot{z} = -z - z^3$.

c. No, since the matrix $\begin{pmatrix} g(x_0) & ad_f g(x_0) & ad_f^2 g(x_0) \end{pmatrix}$ has only rank 2 for $x_0 = 0$.