KTH Matematik

# Solution to Final Exam of SF2842 Geometric Control Theory

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Allowed written material: the lecture notes, the exercise notes, your own classnotes and $\beta$ mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

*Preliminary grades:* 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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**1.** Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system that is both controllable and observable.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.

If $V^* \neq 0$, then system (1) can be made unobservable by some $u = Fx$. . .(4p)

**Answer:** True, since $V^* \subset ker\ C$ and is $A + BF$-invariant for any friend $F$.

(b) If the system (1) has relative degree $(r_1, \cdots, r_m)$, then the only reachability (controllability) subspaces the system has are $\{0\}$ and $R^n$. . . . . . . . . . . . . . . (4p)

**Answer:** False, since controllability subspaces have nothing to do with outputs. One can easily construct a counter example.

(c) Consider an exo-system $\dot{w} = Sw, y_r = qw$, where $dimension\ (w) < dimension\ (x)$, all the eigenvalues of $S$ have non-negative real parts and the pair $(q, S)$ is observable. If we define $e = y - qw$ in system (1) and assume $m = 1$, then there are output matrices $C$ (provided $A$ and $B$ are fixed) such that the full information output regulation problem is always solvable, namely solvable for all exo-systems satisfying the assumptions. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (4p)

**Answer:** True according to discussions in Chapter 6.

(d) Consider

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i,$$

where $x \in N(0) \subset R^n$ and $m < n$. Suppose $\omega_i(x), i = 1, \cdots, n-m$ are row vectors that are linearly independent on $N(0)$ such that $\omega_i(x)g_j(x) = 0, \forall i, j$. If for some $i^* \leq n-m$, $\frac{\partial z(x)}{\partial x} = \omega_{i^*}(x)$, where $z(x)$ is a scalar function, then the system is not controllable in some neighborhood of the origin. .................... (4p)

**Answer:** True, since this would imply $z(x(t)) = z(x(0))$.

(e) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where $x \in R^n$, $f, g, h \in C^\infty$ and $f(0) = 0,\ h(0) = 0$. If the system has relative degree $n$ around the origin, then the system is exponentially stabilizable by a state feedback. .................... (4p)

**Answer:** True, since the system is exactly linearizable (controllable as a linear system) in this case.

**2.** Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= (1\ 0\ 0\ 0)x. \end{aligned}$$

(a) Find $V^*$. .................... (8p)

**Answer:** $V^* = \{x : x_1 = 0,\ x_2 = 0\}$.

(b) Can we find a $u(t)$ such that the corresponding trajectory from $x(0) = 0$ to $x(t_1) = (0\ 0\ 1\ 1)^T$ lies completely in $V^*$? .................... (6p)

**Answer:** Yes, since $x(t_1) = (0\ 0\ 1\ 1)^T$ lies in $R^*$.

(c) Can we find a friend $F$ of $V^*$, such that $A + BF$ has all the eigenvalues with negative real parts? .................... (6p)

**Answer:** Yes. For example, $u_1 = -x_3,\ u_2 = -x_3 - x_4$.

**3.** Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & -1 \\ \alpha_1 & 1 \\ 0 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x. \end{aligned}$$

Design real coefficients $\alpha_1, \alpha_2$ such that the following two conditions are both satisfied simultaneously

(a) the noninteracting control problem is solvable, ........................ (10p) and

(b) the transmission zero(s) is (are all) with a real part of $-2$. ............ (10p)

**Answer:** $\alpha_1 \neq 0,\ \alpha_2 = -\frac{1}{4}$.

**4.** Consider a control system subject to disturbance:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2 + x_3 + u + w_1 \\
\dot{x}_3 &= \alpha x_3 + u \\
y &= x_1,
\end{aligned}$$

where $w_1$ is an unknown nonzero constant (disturbance) and $\alpha$ is a real constant. *Note that in this problem you are allowed to identify stabilizability with controllability and detectability with observability.*

(a) Is the disturbance decoupling problem (DDP) solvable? ................ (4p)
**Answer:** Obviously no.

(b) When $u = 0$, for what $\alpha$ the state trajectory $x(t)$ is always bounded? .... (6p)
**Answer:** $\alpha < 0$.

(c) Show that for almost all values of $\alpha$ the error feedback output regulation problem is solvable if we choose $y_r = 0$ as the reference output (you do not need to design the controller). What is (are) the value(s) of $\alpha$ such that this problem is not solvable? ........................................................ (10p)
**Answer:** $\alpha \neq 1$ (transmission zero), $\alpha \neq 0$ (observability).

**5.** Consider in a neighborhood $N$ of the origin

$$\begin{aligned}
\dot{x}_1 &= (2+\alpha)x_1 + x_1^3 + x_2 \\
\dot{x}_2 &= \alpha x_1^3 - x_2 + x_3 \\
\dot{x}_3 &= f(x_1, x_2, x_3) + u \\
y &= x_3,
\end{aligned}$$

where $\alpha$ is a constant and $f$ is $C^\infty$ with $f(0,0,0) = 0$.

(a) Convert the system into the normal form. .............................. (4p)
**Answer:** Omitted.

(b) Analyze the stability of the zero dynamics in terms of $\alpha$. ............... (6p)
**Answer:** $\alpha > -2$ unstable, $\alpha < -2$ exponentially stable, $\alpha = -2$ asymptotically stable.

(c) When can we use a high gain output feedback control to stabilize the nonlinear system? ........................................................ (4p)
**Answer:** $\alpha < -2$

(d) Is the system without the output exactly linearizable around the origin? . (6p)
**Answer:** Yes.