KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

June 9 2017

Examiner: Xiaoming Hu, phone 790 71 80, mobile 070-796 78 31.

Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

Consider first a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.

(a) If system (1) is controllable and observable, then there exists always a friend F of V^* such that $A + BF$ is a stable matrix. .... (4p)

Answer: False. For example, for a non-minimum phase SISO system.

(b) Let B have full column rank. If the only reachability subspace contained in $ker\ C$ is $\{0\}$, then the system is invertible. .... (4p)

Answer: True, since this implies $R^* = \{0\}$, thus $V^* \cap Im\ B = \{0\}$.

(c) Let $p = m$. If system (1) has relative degree $(r_1, \cdots, r_m)$ and the zero dynamics is unstable, then the system can not be stabilized by any state feedback control $u = Fx$. .... (4p)

Answer: False, since the system can still be controllable.

(d) Consider a controllable and observable (when w is set to 0) linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ \dot{w} &= Sw \\ y &= Cx, \end{aligned}$$

where w is disturbance. If $Im\ E$ is not contained in V^*, then the full information output regulation problem (here $y_r = 0$ is the reference output) is never solvable. (4p)

Answer: False. One can find counter example in the compendium.

(e) Consider a nonlinear single-input system

$$\dot{x} = f(x) + g(x)u$$

where $x \in R^n$, $f, g \in C^\infty$ and $f(0) = 0$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$.

A necessary condition for the system to be exactly linearizable around $x = 0$ is that the rank of matrix A is greater or equal to $n - 1$. .... (4p)

Answer: True, since the linearized system must be controllable and the rank condition is a necessary condition for controllability.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 0 & 0 & -1 \\ 0 & -2 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} u \\ y &= (1\ 1\ 0\ 0)x. \end{aligned}$$

(a) Compute $\mathcal{V}^*$ and express all friends F of $\mathcal{V}^*$............................. (6p)

Answer: $\mathcal{V}^* = \{x \in R^4 : x_1 + x_2 = 0, x_3 + x_4 = 0\}$, $f_{22} - f_{21} = 1,\ f_{24} - f_{23} = \frac{1}{2}$.

(b) Compute $\mathcal{R}^*$ that is contained in $ker\ C$................................ (4p)

Answer: $\mathcal{R}^* = \mathcal{V}^*$.

(c) Can we find a friend F of $\mathcal{V}^*$ such that $(A + BF)$ has all eigenvalues with negative real parts? Justify your answer. .... (6p)

Answer: Yes.

(d) Let $A_F = A + BF$, and $\Omega_F = (C^T, A_F^T C^T, \cdots, (A_F^3)^T C^T)^T$. We formulate the following optimization problem:

$$\min_{F \in R^{2\times 4}} \ rank\ \Omega_F$$

What is the optimal value of $rank\ \Omega_F$? What are the optimal solutions F^*? (4p)

Answer: The dimension of $ker\ \Omega_F$ is maximized when F is a friend of $\mathcal{V}^*$ and in this case $ker\ \Omega_F = \mathcal{V}^*$. Thus the optimal value of rank Ω_F is 2.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x. \end{aligned}$$

(a) What is the zero dynamics? ....(5p)

Answer: $\dot{z}_1 = -4z_1$.

(b) Is the noninteracting problem solvable? .... (5p)

Answer: Yes, since it has a relative degree $(1, 2)$.

(c) Now suppose one of the two output sensors is out of order, namely we have only one output for the system y_1 available. What are the transmission zero(s) now? .... (10p)

Answer: -4. From a) we know

$$\begin{pmatrix} sI - A & B \\ -c_1 & 0 \\ -c_2 & 0 \end{pmatrix}$$

does not have full rank only at $s = -4$. So by taking the last row away from the matrix, the new matrix can only possibly have a rank deficiency at $s = -4$. Pluging in $s = -4$ to the matrix and it does.

4. Consider:

$$\begin{aligned} \dot{x} &= Ax + bu + pqw \\ \dot{w} &= \Gamma w \\ y &= cx, \end{aligned}$$

where α is a positive constant, w represents both disturbances and reference output signals.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -4 & 0 \\ 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad p = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

Some facts you can use: $eig(A) = \{-0.82 \pm 1.8i, -1.68 \pm 0.87i\}$, (A, p) is controllable.

(a) Show the **existence** of an invariant subspace $x = \Pi(\alpha)w$ for the system when u is set to zero (no exact Π is needed) . ... (5p)

Answer: Since A is stable and Γ is antistable, there exists such an invariant subspace.

(b) Show for almost all values of α, the reduced system

$$\begin{aligned} \dot{w} &= \Gamma w \\ y &= c\Pi(\alpha)w \end{aligned}$$

is observable, where both Γ and c are defined as above. What is the value of α that makes the reduced system unobservable?........................... (8p)

Answer: The reduced system is observable if no eigenvalue of Γ is a transmission zero of $\dot{x} = Ax + pu,\ y = cx$, since (A, p, c) is minimal. The transmission zeros are 2 and -1. Therefore, for $\alpha = 2$, the reduced system is not observable.

(c) For $\alpha = 2$, solve the full information output regulation problem (find u), where the tracking error is

$$e = y - (0\ 1\ 0)w.$$

.... (7p)

Answer: (A, b) is stabilizable since A is already stable. The system without w can be converted into the normal form with relative degree 3. Therefore only $z_1 = \pi_1 w$ needs to be solved by a Sylvester equation. Note: although 2 is a transmission zero for (A, p, c), it is not a transmission zero for (A, b, c).

5. Consider in a neighborhood N of the origin

$$\begin{aligned} \dot{x}_1 &= x_1^5 - \alpha x_1 e^{x_3} \sin(x_2) \\ \dot{x}_2 &= x_1^2 - \sin(x_2) + x_3 - e^{x_3} \sin(x_2) u \\ \dot{x}_3 &= e^{x_3} \cos(x_2) u \\ y &= x_3, \end{aligned}$$

where α is a real constant.

(a) Convert the system into the normal form (**Note**: after the new coordinates are chosen, you only need to write down the right hand side of the normal form so explicit that the zero dynamics can be determined). .... (6p)

Answer: $z_1 = x_1,\ z_2 = e^{x_3} \sin(x_2),\ \xi = x_3$....

(b) Analyze the stability of the zero dynamics in terms of α. .... (5p)

Answer: The zero dynamics is

$$\begin{aligned} \dot{z}_1 &= z_1^5 - \alpha z_1 z_2 \\ \dot{z}_2 &= (z_1^2 - z_2)\sqrt{1 - z_2^2}. \end{aligned}$$

Asymptotically stable if $\alpha > 0$ otherwise unstable.

(c) Design a feedback control to stabilize the system for the case when the zero dynamics is asymptotically stable. .... (4p)

Answer: omitted.

(d) Is the system without the output exactly linearizable? .... (5p)

Answer: No, since the linearized system is not controllable.