KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

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Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.

If the system has zero and pole (eigenvalues of A) cancellation, then it can never be minimal (both controllable and observable). .... (4p)

Answer: False. This is only true for the SISO case.

(b) Consider again system (1) and assume it has relative degree $(r_1, \cdots, r_m)$, then $R^* = \{0\}$. .... (4p)

Answer: True.

(c) Consider the following system in which the state is $(x^T, w^T)^T$.

$$\begin{aligned} \dot{x} &= Ax + Bw \\ \dot{w} &= Sw \\ y &= Cx, \end{aligned} \tag{2}$$

where $x \in R^n$, $w \in R^m$. If (A, B) is controllable and (C, A) is observable, then system (2) is also observable. .... (4p)

Answer: False. We need to consider the eigenvalues of S and zeros of the (A, B, C) system.

(d) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

where $x \in R^n$, $f, g, h \in C^\infty$ and $f(0) = 0,\ h(0) = 0$.
The following is the linearized approximation of (3):

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where $A = \frac{\partial f(x)}{\partial x}|_{x=0},\ b = g(0),\ c = \frac{\partial h(x)}{\partial x}|_{x=0}$.
If (A, b) is controllable, then system (3) is exactly linearizable around the origin. (4p)

Answer: False, since we also need certain distribution to be involutive.

(e) Consider again system (3). If (c, A) is not observable, then system (3) can not have relative degree n around the origin. .... (4p)

Answer: True, since relative degree n would imply observability.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x. \end{aligned}$$

(a) Find V^*. .... (6p)

Answer: $V^* = \{x \in R^4, x_1 = x_2 = 0\}$

(b) Find R^*. .... (4p)

Answer: $R^* = \{x \in R^4, x_1 = x_2 = x_3 = 0\}$

(c) Discuss if we can find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? .... (5p)

Answer: No

(d) Find a friend F of V^* that makes V^* attractive, namely for any solution $x(t)$ of the closed-loop system $\dot{x} = (A + BF)x$, the Euclidean distance from $x(t)$ to V^* tends to zero as $t \to \infty$. .... (5p)

Answer: Yes, if we choose $u_1 = -9x_1 + 4x_2 - x_3$ for example.

3. Consider the system

$$\dot{x} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 2 & 1 \\ \alpha & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x,$$

where α is a real constant.

(a) For what value of α is the noninteracting control problem solvable? ..... (6p)
Answer: $\alpha \neq 2$.

(b) What is the transmission zero(s) of the system when the noninteracting control problem is solvable? .... (6p)
Answer: $s = -2$.

(c) Suppose now the first output y_1 is taken away from the system, namely only y_2 is kept as output. What is the transmission zero(s) of the system now? (8p)
Answer: Case 1. $\alpha = 2$. We can treat $2u_1 + u_2$ as one control, then the corresponding SISO system has relative degree 3, which gives $s = -3, -2, 3$. Case 2. $\alpha \neq 2$. Then the only possible zero would be $s = -2$, and this is verified by checking the rank of the system matrix at $s = -2$.

4. Consider:

$$\dot{x} = Ax + b(u + w(t))$$

where $x \in R^n$, $u \in R$, A is a stable matrix, i.e., all eigenvalues have negative real parts, and (A, b) is controllable.

(a) Let $w(t) = \alpha + \beta \sin(\omega t + \phi)$, where α, $\beta > 0$, $\omega > 0$, ϕ are arbitrary real constants. What is the minimum order of the system such that there exists an output $y = cx$ that reconstructs $w(t)$ in stationarity when $u = 0$? ....(5p)
Answer: three.

(b) for any c such that cx reconstructs $w(t)$ in stationarity when $u = 0$, is (c, A) observable? .... (3p)
Answer: Yes. The proof is in the compendium.

(c) Now let $n \geq 3$, $y = cx$ such that $det(sI - A) \cdot c(sI - A)^{-1}b = rs^2 + p$, $w(t)$ is a constant disturbance, discuss conditions on r and p ($r^2 + p^2 \neq 0$) such that the full information output regulation problem with the tracking error $e = y - \sin(t)$ is solvable (*a solution is not required*). ....(6p)
Answer: In this case, the exo system has eigenvalues $0, \pm j$. Thus, the roots of $rs^2 + p$, which are zeros, can not overlap with any of the eigenvalues. The rest is omitted.

(d) For $n = 3,\ r = 0,\ p = 1$, can we solve the error feedback output regulation problem for the system specified in (c)? . . . . (6p)

Answer: Yes, since the full information problem is solvable and the expanded system is observable.

5. Consider in a neighborhood N of the origin

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_2\\ \dot{x}_2 &= \alpha x_1^3 - x_2 + x_3 + \beta u\\ \dot{x}_3 &= -x_1^5 + u\\ y &= x_3,\end{aligned}$$

where $\alpha,\ \beta$ are constant.

(a) Convert the system into the normal form. (Hint: no need to start with one-forms). . . . . (5p)

Answer: $\xi = x_3$ and we can take $z_1 = x_1, z_2 = x_2 - \beta x_3$. The rest is omitted.

(b) Analyze the stability of the zero dynamics in terms of $\alpha,\ \beta$. . . . . (5p)

Answer: $\dot{z}_1 = z_1^3 + z_2, \dot{z}_2 = \alpha z_1^3 - z_2 + \beta z_1^5$. Thus, asymp. stable if $\alpha < -1$, unstable if $\alpha > -1$. When $\alpha = -1$, asymp. stable if $\beta < 0$, unstable if $\beta > 0$, stable if $\beta = 0$.

(c) When $\beta \neq 1$, show that the nonlinear system can be asymptotically stabilized for any α. . . . . (5p)

Answer: When $\beta \neq 1$, the linearized part is controllable, thus, by principle of stability by first approximation, the nonlinear system can be stabilized by a linear feedback control.

(d) Show the system without the output is exactly linearizable when $\beta = 0$. . . (5p)

Answer: Omitted.