KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

14.00-19.00, March 12, 2019

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Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (1)$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.
If system (1) is observable, then $V^* = 0$. .... (4p)
Answer: False, for example, as long as the system has relative degrees and nontrivial zero dynamics, V^* is not zero.

(b) If system (1) does not have any relative degree $(r_1, \cdots, r_m)$, then it does not have any (transmission) zero. .... (4p)
Answer: False. In general, as long as R^* is strictly contained in V^*, there would be zeros.

(c) Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ \dot{w} &= Sw \\ y &= Cx, \end{aligned}$$

where w is disturbance, (A, B) is controllable, and no eigenvalue of S has negative real part. If $Im\ E$ is contained in V^*, then the full information output regulation problem (here $y_r = 0$ is the reference output) is solvable. ..... (4p)
Answer: False. DDP does not in general guarantee that there is friend F of V^* such that $A + BF$ is stable. For output regulation we need to check if 0 is a zero of the system.

(d) Consider

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i,$$

where $x \in R^n$ and $m < n$. If the system is controllable then the distribution $span\{g_1(x), ..., g_m(x)\}$ is not involutive. . . . . (4p)

Answer: True. This is necessary for controllability.

(e) Consider a nonlinear single-input system

$$\dot{x} = f(x) + g(x)u$$

where $x \in R^n$, $f, g \in C^\infty$ and $f(0) = 0$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$.

A necessary condition for the existence of an output mapping $h(x)$ such that the system has a relative degree r at the origin is that $rank\ A \geq r - 1$. . . (4p)

Answer: True. We can see this easily in the normal form.

2. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} u \\ y &= (0\ 1\ 0\ 0)x.\end{aligned}$$

(a) Find V^*. . . . . (8p)

Answer: $V^* = \{x : x_1 = x_2 = 0\}$.

(b) Can we find a $u(t)$ such that the corresponding trajectory from $x(0) = \mathbf{0}$ to $x(t_1) = (0\ 0\ 2\ 1)^T$ lies completely in V^*? . . . . (6p)

Answer: Yes, since $R^* = V^*$.

(c) Can we find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? . . . . (6p)

Answer: Yes.

3. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \alpha & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x,\end{aligned}$$

where α is a real constant.

(a) For what value of α is the noninteracting control problem solvable? . . . . . (4p)

Answer: $\alpha \neq 1$.

(b) What is the (transmission) zero(s) of the system when the system has relative degrees? .... (6p)

Answer: $\frac{1-2\alpha}{1-\alpha}$.

(c) Suppose now the second input u_2 is taken away from the system, namely only u_1 is kept. What is the (transmission) zero(s) of the system now? ...... (10p)

Answer: $s_0 = 1$ when $\alpha = 0$, there are no zeros for other α. This can be done either by computing the rank of the system matrix directly, or by taking either y_1 or y_2 as output and computing the zeros for the respective siso systems. s_0 being a zero implies that it must be a common zero of the two siso systems. Further it must be $\frac{1-2\alpha}{1-\alpha}$ when $\alpha \neq 1$. This leads to the conclusion that we only need to check the rank of the system matrix for $\alpha = 0$ $(s = 1)$ and $\alpha = \frac{2}{3}$ $(s = -1)$.

4. Consider:

$$\begin{aligned} \dot{x} &= Ax + bu + Pw \\ \dot{w} &= \Gamma w \\ y &= cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 4 & 0 \\ -1 & -\alpha & -1 \\ 0 & 0 & -\alpha \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

and α is a **positive constant**.

(a) Show the **existence** of an invariant subspace $x = \Pi(\alpha)w$ for the system when u is set to zero (no exact Π is needed) . ... (5p)

Answer: We only need to verify that A is a stable matrix.

(b) Find the values of α, such that the reduced system

$$\begin{aligned} \dot{w} &= \Gamma w \\ y &= c\Pi(\alpha)w \end{aligned}$$

is observable. .... (7p)

Answer: $Pw = \bar{b}qw$, where $\bar{b} = [1\ 0\ 1]^T$ and $q = [1\ 0]$. We can easily verify (c, A) and (q, Γ) are observable. Then we compute the zeros for $(A, \bar{b}, c)$, which are $-\alpha - 2$ and $-\alpha + 2$. This leads the conclusion that $\alpha \neq 1,\ 2$.

(c) Show for almost all values of α, the full information output regulation problem with the tracking error $e = y - w_2$ is solvable (*a solution is not required*) . What are the values of α that make the problem unsolvable? .... (8p)

Answer: We first compute the zero for (A, b, c), which is $2 - \alpha$. We can verify that $\alpha = 1,\ 2$ make the Sylvester equation unsolvable.

5. Consider in a neighborhood N of the origin

$$\begin{aligned}\dot{x}_1 &= x_2^3 + e^{x_3}u \\ \dot{x}_2 &= -x_2 + \alpha x_3^3 + 2x_2^4 \\ \dot{x}_3 &= x_2^3 + x_3^9 + 2e^{x_3}u \\ y &= x_1,\end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form. .... (6p)

Answer: the system has relative degree 1. $\xi = x_1,\ z_1 = x_2, z_2 = x_3 - 2x_1$. The rest is omitted.

(b) Analyze the stability of the zero dynamics in terms of α. .... (7p)

Answer: The zero dynamics is: $\dot{z}_1 = -z_1 + \alpha z_2^3 + 2z_1^4,\ \dot{z}_2 = -z_1^3 + z_2^9$. Using center manifold theory, we can show $\alpha \leq 1$: unstable; $\alpha > 1$: asymptotically stable.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. .... (3p)

Answer: omitted.

(d) Is the system without the output exactly linearizable? .... (4p)

Answer: Since the linearized system is not controllable, the system is not exactly linearizable.