KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

14.00-19.00, March 15 2022

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Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 40 points give grade E, and the other grade limits can be found on Canvas.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.
If $\mathcal{V}^* = 0$, then (C, A) is observable. .... (5p)
Answer: True, since $\mathcal{V}^*$ is the maximal unobservable subspace.

(b) Assume that $m = p$. If $\mathcal{R}^* = 0$, then system (1) must have some relative degree $(r_1, \cdots, r_m)$. .... (5p)
Answer: False, since having relative degree is only sufficient for $\mathcal{R}^* = 0$.

(c) Consider an exo-system $\dot{w} = Sw$, where $w \in R^p$ and all the eigenvalues of S have non-negative real part. For system (1) let $m = p = 1$. For a give reference signal $u_r = qw(t)$ Let $u = Fx + qw$. Then we can find C and F such that $Cx = qw$ as $t \to \infty$ if (A, B) is controllable and (q, S) is observable. ..... (5p)
Answer: True according to the discussions in Chapter 6.

(d) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2}$$

where $x \in R^n$, $f, g, h \in C^\infty$ and $f(0) = 0,\ h(0) = 0$.

The following is the linearized approximation of (2):

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \tag{3}$$

where $A = \frac{\partial f(x)}{\partial x}|_{x=0}$, $b = g(0)$, $c = \frac{\partial h(x)}{\partial x}|_{x=0}$. Assume system (2) has relative degree r at the origin and is minimum phase. Then the linearized system (3) is also minimum phase. . . . . (5p)

Answer: False, since the zero dynamics of the nonlinear system may only be asymptotically stable, but not exponentially stable.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 2 & 0 & \alpha^2 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= (1\ 0\ 0\ 0)x, \end{aligned}$$

where α is a real constant.

(a) Find $\mathcal{V}^*$. . . . . (8p)
Answer: $\mathcal{V}^* = \{x : x_1 = 0, x_2 = 0\}$

(b) Find $\mathcal{R}^*$. . . . . (6p)
Answer: $\mathcal{R}^* = \mathcal{V}^*$ if $\alpha^2 \neq 1$, otherwise $\mathcal{R}^* = \{x : x_1 = 0, x_2 = 0, x_3 = x_4\}$

(c) Can we find a friend F of $\mathcal{V}^*$, such that $A + BF$ has all the eigenvalues with negative real parts? . . . . (6p)
Answer: Yes if $\alpha^2 \neq 1$.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ \alpha & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} x, \end{aligned}$$

where α is a real constant.

(a) For what value of α is the noninteracting control problem solvable? . . . . . (6p)
Answer: $\alpha \neq -1$.

(b) What is the transmission zero(s) of the system when the noninteracting control problem is solvable? . . . . (7p)
Answer: -1.

(c) Suppose now the first output y_1 is taken away from the system, namely only y_2 is kept as output, and $\alpha \neq -1$. What is the transmission zero(s) of the system now? .... (7p)

Answer: The only possible zero in this case is -1. We let $s = -1$ and compute the rank of the system matrix, which shows that $s = -1$ is indeed a zero.

4.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2 - x_3 + u + w_1 \\ \dot{x}_3 &= \alpha x_3 - 2u \\ y &= x_1, \end{aligned}$$

where w_1 is an unknown nonzero constant (disturbance).

(a) Is the disturbance decoupling problem (DDP) solvable? .... (4p)

answer: No, since the disturbance channel is not in $\mathcal{V}^*$.

(b) When $u = 0$, show that if $\alpha < 0$, then for any given initial condition, the output converges to a constant as $t \to \infty$. .... (7p)

answer: This can be shown by letting $\bar{x}_1 = x_1 - w_1$ and showing the system is asymptotically stable under the new state dynamics.

(c) Show that for almost all values of α the error feedback output regulation problem is solvable if we choose $y_r = 0$ as the reference output (you do not need to design the controller). What is (are) the value(s) of α such that this problem is not solvable? .... (9p)

answer: Since $\dot{w}_1 = 0$, and the zero is $\alpha - 2$, $\alpha \neq 2$. $\alpha \neq 0$ either, otherwise the augmented system is observable. We also need to check for what α the system may not be controllable.

5. Consider in a neighborhood N of the origin

$$\begin{aligned} \dot{x}_1 &= x_2^3 + e^{x_3} u \\ \dot{x}_2 &= -x_2 + \alpha x_3^3 + 2x_2^4 \\ \dot{x}_3 &= x_2^3 + x_3^9 + 2e^{x_3} u \\ y &= x_1, \end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form. .... (6p)

Answer: the system has relative degree 1. $\xi = x_1,\ z_1 = x_2, z_2 = x_3 - 2x_1$. The rest is omitted.

(b) Analyze the stability of the zero dynamics in terms of α. .... (7p)

Answer: The zero dynamics is: $\dot{z}_1 = -z_1 + \alpha z_2^3 + 2z_1^4$, $\dot{z}_2 = -z_1^3 + z_2^9$. Using center manifold theory, we can show $\alpha \leq 1$: unstable; $\alpha > 1$: asymptotically stable.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. .... (3p)

Answer: omitted.

(d) Is the system without the output exactly linearizable? .... (4p)

Answer: Since the linearized system is not controllable, the system is not exactly linearizable.

Good luck!