KTH Matematik

# Exam in SF2862 Stochastic Decision Support Models.
## Monday June 8, 2009, 14.00–19.00

*Examiner:* Krister Svanberg, tel. 790 71 37

*Allowed tools:* Pen/pencil, ruler and eraser. A formula-sheet is handed out.
No calculator is allowed!

*Solution methods:* Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Conclusions should be motivated. If you use methods other than what has been taught in the course, you must explain very carefully.

*Note!* Personal number must be written on the title page. Write only one excercise per sheet. Number the pages and write your name on each sheet!
25 points, including bonus points from the homeworks, are sufficent for passing the exam. For 23-24 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam have been reported. Contact the examiner as soon as possible for such a completion.

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**1.** The company ABC needs exactly 100 kg per week of a certain expensive metal during the forthcoming 4 weeks. ABC may order the metal (from its metal supplier) only on Friday evenings. The ordered quantity (if any) is then delivered early Monday morning, in time for the production activities of the week. The unit price for the metal is 1000 Euro/kg. In addition, there is a fixed ordering cost of 700 Euro each time an order is placed (for administration, transportation, etc.). If metal is stored from one week to another at ABC, the holding cost is 2 Euro per kg and week. On the Friday evening just before the first week of the considered planning period, the inventory is empty, and it should be empty also by the end of the fourth week.

(a) Calculate an optimal order plan for the company ABC. Use a systematic method which can be used in practice also if there were a much larger number of weeks to consider. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (7p)

(b) Assume that the ordering cost is increased from 700 to $700 + c$, where $c$ is a positive constant. What is the largest value of $c$ for which the optimal plan from (a) above is still optimal? Call this largest value $c_{\text{max}}$. What is the (new) optimal order plan if $c$ is slightly larger than $c_{\text{max}}$? . . . . . . . . . . . . . . . . . . . . . . (3p)

**2.** (a) On the formula-sheet, there are some formulas for M/M/1. When deducing the formula for $L$ from the formulas for $P_0$ and $P_n$, one uses the identity
$\sum_{n=1}^{\infty} n\rho^n = \frac{\rho}{(1-\rho)^2}$ , for $|\,\rho\,| < 1$. Prove this identity! . . . . . . . . . . . . . . . . . . . . (2p)

(b) On the formula-sheet, there are also some formulas for M/M/2, in particular $P_0 = \dfrac{1-\rho}{1+\rho}$, $P_n = 2\rho^n P_0$ for $n \geq 1$, and $L = \dfrac{2\rho}{1-\rho^2}$, where $\rho = \lambda/(2\mu) < 1$. Verify that these formulas hold, by formulating and solving the relevant balance equations and using the definition of $L$. ................................ (4p)

(c) Consider now a system M/M/3.
Calculate $P_n$ and $L$ for the special case that $\lambda = \mu = 1$. ................. (4p)

**3.** This exercise deals with an optimal serving strategy for the famous tennis player "Monty" Mulligan (MM) against his persistent opponent George Johnson (GJ).
When MM serves, he gets two chances to serve in bounds. If he twice fails to serve in bounds, he loses the point. (This is standard rules in tennis.)
If he makes a "hard" serve, he serves in bounds with probability $p$.
If he makes a "lob" serve, he serves in bounds with probability $q$.
If his hard serve is in bounds, he wins the point with probability 3/4.
If his lob serve is in bounds, he wins the point with probability 1/2.
Here, $p$ and $q$ are given numbes which satisfy $0 < p < q \leq 1$.

If the cost is $+1$ for each point lost and $-1$ for each point won, the considered problem is to determine the optimal serving strategy to minimize the expected cost per point.

(a) Assume that $p = 1/2$ and $q = 7/8$.
Determine the optimal serving strategy. ................................ (5p)

(b) Are there any values on $p$ and $q$, with $0 < p < q \leq 1$, for which the optimal strategy is to first make a lob serve and then (if this first serve was not in bounds) make a hard serve. Motivate your answer mathematically. ...... (5p)

**4.** A queueing system consist of two service facilities, called $F_1$ and $F_2$.
In each of the facilities there is a single server with exponentially distributed service time with mean 6 minutes.

Customers arrive to the system according to a poisson process with rate 9 customers per hour. Each such new customer will either go to facility $F_1$, with probablity $p$, or to facility $F_2$, with probablity $1-p$. The value of this parameter $p \in [0, 1]$ can be chosen by the manager of the system, which means that she can (in some sense) steer the arriving customers.

A customer who has been served at $F_1$ will either go to $F_2$, with probability 0.5 (regardless of how many times he has already been at $F_2$), or immediatly leave the whole system, with probability 0.5. A customer who has been served at $F_2$ will either go to $F_1$, with probability 0.2 (regardless of how many times he has already been at $F_1$), or immediatly leave the whole system, with probability 0.8.

(a) For which $p \in [0, 1]$ can the system be in steady state? .................. (2p)

(b) Assume that the manager wants to choose $p$ such that the expected number of customers in the system (in steady state) is as small as possible.
Calculate such an optimal $p$. ........................................... (4p)

(c) Assume now that the manager instead wants to choose $p$ such that the steady state probability that the system is empty is as large as possible.
Calculate such an optimal $p$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (3p)

(d) Assume finally the manager instead wants to choose $p$ such that the expected total time in the system for a customer (which enter the system in steady state) is as small as possible. Calculate such an optimal $p$. . . . . . . . . . . . . . . . . . . . . (3p)

**5.** **Note:** At a first glimpse, this final exercise may look almost identical to one of the exercises on the previous exam (March 2009), but there is an important difference in the objective function! This time, the *maximal* number of balance trials (in the worst case) should be minimized, instead of the *expected* number of balance trials.

On the round table in front of Captain Hook, there are 9 seemingly identical coins. He happens to know that exactly one of them is false, and he wants to find out which one. The false coin has lower weight than the others, and to Hook's disposal is a balance with two bowls. If he puts $k$ coins in each bowl, the balance will tell him if the two bowls contain equal weight or if one of the bowls contains lower weight than the other, and in that case which one.

Hook has a reputation of beeing efficient and pessimistic, so he wants to use a strategy which minimizes the *maximal* number of balance trials (in the worst case).

Let $V(n)$ be the minimal number of balance trials, *in worst case*, needed to identify the false coin when it is known to be among $n$ specific coins and an optimal strategy is used. In particular $V(1) = 0$.

(a) Formulate a recursive equation for $V(n)$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (3p)

(b) Use this recursive equation to calculate $V(n)$ for $n = 2, 3, 4, 5, 6, 7, 8, 9$. . . (4p)

(c) Explain the optimal strategy for Captain Hook. . . . . . . . . . . . . . . . . . . . . . . . . . (1p)

Good luck!