Arbitrage Bounds for Volatility Derivatives as Free Boundary Problem

Bruno Dupire
Bloomberg L.P. NY

bdupire@bloomberg.net
Variance Swaps

- Vanilla options are complex bets on $S/\sigma$
- Variance Swaps capture volatility independently of $S$
- Payoff: $\sum (\ln \frac{S_{t_{i+1}}}{S_{t_i}})^2 - VS_0$
  \begin{align*}
    \text{Realized Variance}
  \end{align*}
- Replicable from Vanilla option (if no jump):
  \begin{align*}
    \ln S_T &= \ln S_0 + \int_0^T \frac{dS}{S} + \frac{1}{2} \int_0^T (d\ln S)^2
  \end{align*}
Options on Realized Variance

- Over the past couple of years, massive growth of
  - Call on Realized Variance:
    \[
    \left( \sum (\ln \frac{S_{t_i+1}}{S_{t_i}})^2 - L \right)^+ 
    \]
  - Put on Realized Variance:
    \[
    (L - \sum (\ln \frac{S_{t_i+1}}{S_{t_i}})^2)^+ 
    \]
- Cannot be replicated by Vanilla options
Classical Models

Classical approach:

• To price an option on $X$:
  – Model the dynamics of $X$, in particular its vol
  – perform dynamic hedging

• For options on realized variance:
  – Hypothesis on the vol of VS
  – Dynamic hedge with VS

But Skew contains important information and we will examine how to exploit it to obtain bounds for the option prices.
• Option prices of maturity $T \Leftrightarrow$ Risk Neutral density of $S_T$:
  
  $$\varphi(K) = \frac{\partial^2}{\partial K^2} \mathbb{E}[(S_T - K)^+]$$

• **Skorokhod problem:** For a given probability density function such that $\int x \varphi(x) dx = 0$, $\int x^2 \varphi(x) dx = v < \infty$, find a stopping time $\tau$ of finite expectation such that the density of a Brownian motion $W$ stopped at $\tau$ is $\varphi$

• A continuous martingale $S$ is a time changed Brownian Motion:
  
  $$W_t \equiv S_{\inf\{u: <S>_u > t\}} \text{ is a BM, and } S_t = W_{<S>_t}$$
• \( \tau \) solution of Skorokhod: \( W_\tau \sim \varphi \)

Then \( S_T \equiv W_{\tau \land t/(T-t)} \) satisfies \( S_T = W_\tau \sim \varphi \)

• If \( S_T \sim \varphi \), then \( \tau \equiv < S >_T \) is a solution of Skorokhod

as \( W_\tau = W_{<S>_T} = S_T \sim \varphi \)
• Possibly simplest solution $\tau$: exit time
$B = \{ K, T | (K, T) \in \text{Barrier} \}$

$\varphi = \text{Density of } W_{\tau}$

**PDE:**

\[
\begin{cases}
\frac{\partial \phi}{\partial T}(K, T) = \frac{\partial^2 \phi}{\partial K^2}(K, T) & \text{on } B^C \\
\phi(K, t) = \varphi(K) & \text{on } B
\end{cases}
\]

**BUT:** How about Density $\rightarrow$ Barrier?
• Given \( \varphi \), define \( \overline{f}(K) \equiv \int |x - K| \varphi(x) dx \)

• If \( dX_s = \sigma(X_s, s) dW_s \), \( Cf(K, t) \equiv \mathbb{E}[|X_t - K|] \) satisfies
\[
\frac{\partial f}{\partial t} = \frac{\sigma^2(K, t)}{2} \frac{\partial^2 f}{\partial K^2} \geq 0
\]

• Apply the previous equation with \( \sigma(K, t) = 1 \)
  until \( f(K, t_K) = \overline{f}(K) \).

• Then for \( t > t_K \), \( \sigma(K, t) \equiv 0 \) (\( \Rightarrow f(K, t) = \overline{f}(K) \))
• Define $\tau$ as the hitting time of 

$$B \equiv \{(K, t) : \sigma(K, t) = 0\}, \ X_t = W_{\tau\land t}$$

• Then 

$$f(K, t) \equiv \mathbb{E}[|X_t - K|] = \mathbb{E}[|W_{\tau\land t} - K|]$$

$$f(K, \infty) = \lim_{t \to \infty} f(K, t) = \mathbb{E}[|W_\tau - K|]$$

• Thus $W_\tau \leftrightarrow \varphi$, and $B$ is the ROOT barrier
PDE computation of ROOT (3)

Interpretation within Potential Theory
\[
\min(<S>_T - L)^+ \leftrightarrow \max \mathbb{E}[W_{\delta\wedge L}^2]
\]

- realized Variance \( RV = <S>_T = \tau \)
- Call on \( RV: (\tau - L)^+ \)

\[
\text{Ito: } W_{\tau}^2 = W_{\tau\wedge L}^2 + \int_{\tau\wedge L}^{\tau} W_t dW_t + \tau - \tau \wedge L
\]

Taking expectation, \( v = \mathbb{E}[W_{\tau\wedge L}^2] + \mathbb{E}[(\tau - L)^+] \)

- Minimize one expectation amounts to maximize the other one
Suppose \( dY_t = \sigma_t dW_t \), then define \( f_Y(K, T) \equiv \mathbb{E}[|Y_T - K|] \)

- \( f_Y \) satisfies \( \frac{\partial f_Y}{\partial T} = \frac{1}{2} \mathbb{E}[\sigma_T^2|Y_T = K] \frac{\partial^2 f_Y}{\partial K^2} \) \quad (Dupire 96)

Let \( \tau \) be a stopping time.

- For \( X_t = W_{\tau \land t} \), one has \( \sigma_t = \mathbb{I}_{\tau > t} \) and \( \mathbb{E}[\sigma_T^2|X_T = K] = \mathbb{P}[\tau > T|X_T = K] \)
- \( \Rightarrow dY_t = a(Y_t, t)dW_t \) where \( a(y, t) \equiv \mathbb{P}[\tau > t|X_t = y] \) generates the same prices as \( X \): \( f_Y(K, T) = f_X(K, T) \) for all \( (K, T) \)
- \( \Rightarrow \) For our purpose, \( \tau \) identified by \( a(x, t) = \mathbb{P}[\tau > t|X_t = x] \)
Optimality of ROOT

As $\mathbb{E}[W_{\tau\wedge L}^2] = \int (\mathbb{E}[|W_{\tau\wedge L} - K|] - |W_0 - K|)dK$,

to maximize $\mathbb{E}[|W_{\tau\wedge L} - K|] \implies$ to maximize $\mathbb{E}[W_{\tau\wedge L}^2] \iff$ to minimize $\mathbb{E}[(\tau - L)^+]$

$\mathbb{E}[|W_{\tau\wedge L} - K|] = f(K, L)$ and $a(x, t) = \mathbb{P}[\tau > t | W_t = x]$, $f$ satisfies:

$$\begin{cases} f_2 = \frac{1}{2}a^2(x, t)f_{11} & \text{for } a > 0 \\ f_2 = 0 & \text{for } a = 0 \end{cases}$$

$f$ is maximum for ROOT time, where $a = 1$ in $B^C$ and $a = 0$ in $B$
ROST Filling Scheme
• Given, \( \varphi \), define \( C(K, \infty) \equiv \int |x - K| \varphi(x) \, dx \)

• Now define \( \mu_0(K) = C(K, \infty) - |K| \)

• \( \mu_{t+\delta t} = \min(\mu_t P_{\delta t}, \mu_t) \), where \( \mu_t P_{\delta t}(x) = \int_{-\infty}^{\infty} \mu_t(x + y) \frac{e^{-y^2/(2\delta t)}}{\sqrt{2\pi\delta t}} \, dy \)
• Define \( B(t) = \{ K : \mu_t P_{\delta t}(K) = \mu_t(K) \} \)

• \( B \) is the frontier of the reverse barrier of ROST
- Normal density:

- Asymmetric density:
CONCLUSION

• Skorokhod problem is the right framework to analyze range of exotic prices constrained by Vanilla prices

• Barrier solutions provide canonical mapping of densities into barriers

• They give the range of prices for option on realized variance

• The Root solution diffuses as much as possible until it is constrained
  The Rost solution stops as soon as possible

• We provide explicit construction of these barriers