Some Applications of Classical PDE in finance

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Main aims of this contribution

Joint work with Peter Carr (Bloomberg and Courant) and Tai-Ho Wang, National Chung Cheng U., Chia Yi, Taiwan) using Hadamard’s method and Lie’s symmetry groups in PDE in mathematical finance. To briefly discuss three recent contributions

- Work joint with Peter Carr on multi-asset stochastic local variance
- Work joint with Wang on special solvable quadratic and inverse quadratic interest rate models
- Joint work with both Carr and Wang on generating and characterizing solvable one-dimensional diffusions
Generalization of multi-variance contracts to many assets.

What is a variance contract?

Such a contract specifies a fixed strike \( K > 0 \) and a fixed maturity date \( T \). Just after \( T \) the contract pays

\[
\delta(S_T - K) \, d < S_T > T,
\]

where

\[
d < S_T > T = \text{local quadratic variation}
\]

and where \( \delta(S_T - K) \) is Dirac’s Delta function. For example if \( dS_t = a_t dW_t \),

\[
< dS_T > T = a_T^2 dt.
\]

**Question** What is the natural generalization of this contract in the context of a contract on a basket of assets?

**Answer:** The answer will be related to finding a new contract whose payoff is a cut-off of a fundamental solution for an appropriate **elliptic equation** in \( n \) variables.
Breeden and Litzenberger

\[ C(S_t, t, K, T) = e^{-r(T-t)} E[(S_T - K)^+] \]

where: \[ E[(S_T - K)^+] = \int_{\mathbb{R}^+} (S_T - K)^+ p(S_t, t, s, T) ds, \]

and \( p(s_t, t, s_T, T) \) is the transition probability density for passage from \( S_t \) at time \( t \) to \( S_T \) at time \( T \).

So

\[ C_{KK}(K, T) = e^{-r(T-t)} E[(S_T - K)^+]_{KK} = e^{-r(T-t)} E[\delta_K(S)] \]
\[ = e^{-r(T-t)} p(S_t, t, K, T) \]

ie. Breeden Litzenberger allows us to recover the transition probability from knowledge of the quoted market prices of call options.
Breeden-Litzenberger in $n$-D

Is there a generalization of Breeden and Litzenberger to $n$ assets?

- Answer: A positive answer (kind of) appeared in the case of two assets in the book by Lipton (World Scientific, 2002). The answer involves the Radon transform.
- The generalization to the $n$-asset case is straightforward and also can be formulated in terms of the Radon transform.
- However the answer may be of limited practical interest as it assumes liquid market for index options with a continuum of weights.
- We will return to it later, time permitting.
For simplicity assume zero rates and no dividends. Under the risk neutral measure we have

\[ dS_t = a_t dW_t \]

This is a **stochastic volatility model** in that \( a_t \) is allowed to depend on sources of uncertainty above and beyond the uncertainty in the spot level \( S_t \).

For the quadratic variation we have

\[ d < S >_t = a_t^2 dt \]
Dupire showed that the local variance

\[ E_0 \left[ a_T^2 \mid S_T \in dK \right] \]

can be determined from the initial market value of straddles struck around \( K \) and maturing at \( T \).
Let \( V_0(K, T) \) be the values of such straddles at time 0. Then

\[ E_0 \left[ a_T^2 \mid S_T \in dK \right] = 2 \frac{\partial}{\partial T} V_0(K, T) \frac{\partial^2}{\partial K^2} V_0(K, T) \]

Ratio of a calendar spread of straddles to a butterfly spread of straddles.
To establish his formula, Dupire uses the Tanaka-Meyer theorem to get

\[ |S_T - K| = |S_0 - K| + \int_0^T \text{sgn}(S_t - K) dS_t + \int_0^T \delta_K(S_t) a_t^2 dt \]

Next, take risk-neutral expectations

\[ V_0(K, T) = |S_0 - K| + \int_0^T E_0 \left[ a_T^2 \delta(S_T - K) \right] dt \]

\[
E_0 \left[ a_T^2 \bigg| S_T = K \right] \\
= \frac{E_0 \left[ a_T^2 \delta(S_T - K) \right]}{E_0 \left[ \delta(S_T - K) \right]} \\
= \frac{1}{2} E_0 \left[ a_T^2 \delta(S_T - K) \right] \\
= \frac{\partial^2 V(K, T)}{\partial K^2} \]
(In summary)

\[
\frac{\partial V_0(K, T)}{\partial T} = \frac{1}{2} E_0 \left[ a_T^2 | S_T \in dK \right] \frac{\partial^2}{\partial K^2} V(K, T)
\]

So

\[
E_0 \left[ a_T^2 | S_T \in dK \right] dt = 2 \frac{\partial V_0(K, T)}{\partial T} \frac{\partial^2}{\partial K^2} V(K, T)
\]

Therefore the initial expectation of the terminal local variance given that \( S_T = K \) is twice the ratio of a calendar spread to a butterfly spread.

QED
It is seen that the key element in Dupire’s proof is that

\[
\text{One dimensional Laplacian } F = 2\delta_K(S)
\]

where

\[
F = |S_T - K|
\]

Therefore the key was that function \( |S_T - K| \) is a fundamental solution of the Laplacian in \( 1 - D \).
First case is where the dynamics is normal as opposed to lognormal

\[
dS_{1t} = a_t \sigma_1 dW_{1t}
\]

\[
dS_{2t} = a_2 t \sigma_2 dW_{2t} \quad t \in [0, T]
\]
First case is where the dynamics is normal as opposed to lognormal.

\[ dS_{1t} = a_t \sigma_1 dW_{1t} \]
\[ dS_{2t} = a_{2t} \sigma_2 dW_{2t} \quad t \in [0, T] \]

Here the volatility of the stock \( S_i \) has an idiosyncratic component \( \sigma_i \) and a stochastic component \( a_t \) common to both stocks. \( W_{1t} \) and \( W_{2t} \) are two correlated Brownian motions. The instantaneous variance of \( dS_{1t} + dS_{2t} \) at time \( t \).

\[ a_t^2 (\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2) \]
Generalization to two assets: Normal Dynamics

- First case is where the dynamics is normal as opposed to lognormal

\[ dS_{1t} = a_t \sigma_1 dW_{1t} \]
\[ dS_{2t} = a_{2t} \sigma_2 dW_{2t} \quad t \in [0, T] \]

- Here the volatility of the stock \( S_i \) has an idiosyncratic component \( \sigma_i \) and a stochastic component \( a_t \) common to both stocks. \( W_{1t} \) and \( W_{2t} \) are two correlated Brownian motions. The instantaneous variance of \( dS_{1t} + dS_{2t} \) at time \( t \).

\[ a_t^2 (\sigma_1^2 + 2\rho \sigma_1 \sigma_2 + \sigma_2^2) \]

- The idiosyncratic component of this instantaneous variance is

\[ V_I = \sigma_1^2 + 2\rho \sigma_1 \sigma_2 + \sigma_2^2 \]
Our objective is to synthesize the bivariate local variance $a_t^2 V_I \delta(S_{1t} - K_1, S_{2t} - K_2) dt$. Note that this payoff is zero unless $(S_{1T}, S_{2T} \in (dK_1, dK_2)$ and in this event $a_T^2 V_I dt$ is the increment in the quadratic variation of $S_1 + S_2$. 
Our objective is to synthesize the bivariate local variance \( a_t^2 V_I \delta(S_{1t} - K_1, S_{2t} - K_2) dt \). Note that this payoff is zero unless \((S_{1T}, S_{2T} \in (dK_1, dK_2)\) and in this event \( a_T^2 V_I dt \) is the increment in the quadratic variation of \( S_1 + S_2 \).

Natural candidate to take the place of \(|S - K|\) in the 2D case is (up to a constant and an orthogonal transformation) the fundamental solution of the Laplacian in \( 2 - D \).
Our objective is to synthesize the bivariate local variance $\sigma^2 V_I \delta(S_{1t} - K_1, S_{2t} - K_2)dt$. Note that this payoff is zero unless $(S_{1T}, S_{2T}) \in (dK_1, dK_2)$ and in this event $\sigma^2 V_I dt$ is the increment in the quadratic variation of $S_1 + S_2$.

Natural candidate to take the place of $|S - K|$ in the $2D$ case is (up to a constant and an orthogonal transformation) the fundamental solution of the Laplacian in $2 - D$.

I.e. $g(S_1, S_2)$ satisfies

$$
\frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_1, S_2) + 2 \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 dS_2} g(S_1, S_2) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_1, S_2) = -V_I \delta(S_{1t} - K_1, S_{2t} - K_2)
$$
Normal case ct’d

- Our objective is to synthesize the bivariate local variance \( a_t^2 V_t \delta(S_{1t} - K_1, S_{2t} - K_2) dt \). Note that this payoff is zero unless \( (S_{1T}, S_{2T} \in (dK_1, dK_2) \) and in this event \( a_t^2 V_t dt \) is the increment in the quadratic variation of \( S_1 + S_2 \).

- Natural candidate to take the place of \(|S - K|\) in the 2D case is (up to a constant and an orthogonal transformation) the fundamental solution of the Laplacian in \( 2 - D \).

- I.e. \( g(S_1, S_2) \) satisfies

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\frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_1, S_2) + 2\rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_1, S_2) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_1, S_2) = -V_t \delta(S_{1t} - K_1, S_{2t} - K_2)
\]

- Solution known to be:
Our objective is to synthesize the bivariate local variance \( \alpha_t^2 V_I \delta(S_{1t} - K_1, S_{2t} - K_2) dt \). Note that this payoff is zero unless \((S_{1T}, S_{2T}) \in (dK_1, dK_2)\) and in this event \( \alpha_t^2 V_I dt \) is the increment in the quadratic variation of \( S_1 + S_2 \).

**Natural candidate** to take the place of \(|S - K|\) in the 2D case is (up to a constant and an orthogonal transformation) the fundamental solution of the Laplacian in \( 2 - D \).

I.e., \( g(S_1, S_2) \) satisfies

\[
\frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_1, S_2) + 2\rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_1, S_2) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_1, S_2) = -V_I \delta(S_{1t} - K_1, S_{2t} - K_2)
\]

**Solution** known to be:

\[
g(S_1, S_2) = \frac{1}{\pi \sqrt{(1 - \rho^2) \sigma_1 \sigma_2}} \ln r(S_1, S_2)
\]

\[
r(S_1, S_2) \equiv \left[ \frac{1}{1 - \rho^2} \left[ \left( \frac{S_1 - K_1}{\sigma_1} \right)^2 - 2\rho \frac{S_1 - K_1}{\sigma_1} \frac{S_2 - K_2}{\sigma_2} + \left( \frac{S_2 - K_2}{\sigma_2} \right)^2 \right] \right]
\]
We wish to apply Ito’s formula to $g$ just as we did in the $1 - D$ case for $|S - K|$. However this is not possible due to the singularity of $g$. For now, we illustrate the main idea by proceeding at a formal level. Apply Itô’s formula to the process $G_t$ defined by $G_t = g(S_{1t}, S_{2t})$ implies:

$$g(S_{1T}, S_{2T}) = g(S_{10}, S_{20}) + \int_0^T \left( \frac{\partial}{\partial S_1} g(S_{1t}, S_{2t}) dS_{1t} \right) + \int_0^T \left( \frac{\partial}{\partial S_2} g(S_{1t}, S_{2t}) dS_{2t} \right) + \int_0^T \left( \sum_{i=1}^{2} \frac{\sigma_i^2}{2} \frac{\partial^2}{\partial S_i^2} g(S_{1t}, S_{2t}) \right) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_{1t}, S_{2t}) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_{1t}, S_{2t}) dt$$

Ito’s formula can be used on a truncated version of $g$ we call $g^\epsilon$. 

\[\int_0^T a_t \begin{bmatrix} \frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_{1t}, S_{2t}) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_{1t}, S_{2t}) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_{1t}, S_{2t}) \\ -V_I \delta(S_{1t} - K_1, S_{2t} - K_2) \end{bmatrix} dt\]
We wish to apply Ito’s formula to \( g \) just as we did in the \( 1 - D \) case for \( |S - K| \). However this is not possible due to the singularity of \( g \). For now, we illustrate the main idea by proceeding at a formal level. Apply Itô’s formula to the process \( G_t \) defined by \( G_t = g(S_{1t}, S_{2t}) \) implies:

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g(S_{1T}, S_{2T}) = g(S_{10}, S_{20}) + \int_0^T \frac{\partial}{\partial S_1} g(S_{1t}, S_{2t}) \, dS_{1t} + \int_0^T \frac{\partial}{\partial S_2} g(S_{1t}, S_{2t}) \, dS_{2t} \]

\[
+ \int_0^T a_t^2 \left[ \frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_{1t}, S_{2t}) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_{1t}, S_{2t}) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_{1t}, S_{2t}) \right] \, dt
\]

Ito’s formula can be used on a truncated version of \( g \) we call \( g^\epsilon \).

- Idea: Let \( A = (\Sigma^t \Sigma)^{-1} \) be the inverse of coefficient matrix. Surround singularity \((K_1, K_2)\) by an ellipsoid of radius \( \epsilon \), \( \mathcal{E}_K(\epsilon) = \{X = (x_1, x_2) : X^t AX = \epsilon\} \).
We wish to apply Ito’s formula to \( g \) just as we did in the \( 1 - D \) case for \( |S - K| \). However this is not possible due to the singularity of \( g \). For now, we illustrate the main idea by proceeding at a formal level. Apply Itô’s formula to the process \( G_t \) defined by \( G_t = g(S_{1t}, S_{2t}) \) implies:

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\]

\[
+ \int_0^T \sigma_1^2 \frac{\partial^2}{\partial S_1^2} g(S_{1t}, S_{2t}) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_{1t}, S_{2t}) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_{1t}, S_{2t})
\]

\[
- V_I \delta(S_{1t} - K_1, S_{2t} - K_2)
\]

Ito’s formula can be used on a truncated version of \( g \) we call \( g^\epsilon \).

- Idea: Let \( A = (\Sigma^t \Sigma)^{-1} \) be the inverse of coefficient matrix. Surround singularity \((K_1, K_2)\) by an ellipsoid of radius \( \epsilon \), \( \mathcal{E}_K(\epsilon) = \{X = (x_1, x_2) : X^t A X = \epsilon\} \)

- Glue the solution \( g \) to the solution outside the ball to the solution inside the ball using a quadratic function, while matching both first and second derivatives. Glued together function is called \( g_\epsilon \).
We wish to apply Ito’s formula to \( g \) just as we did in the 1 – \( D \) case for \( |S – K| \). However this is not possible due to the singularity of \( g \). For now, we illustrate the main idea by proceeding at a formal level. Apply Itô’s formula to the process \( G_t \) defined by \( G_t = g(S_{1t}, S_{2t}) \) implies:

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g(S_{1T}, S_{2T}) = g(S_{10}, S_{20}) + \int_0^T \frac{\partial}{\partial S_1} g(S_{1t}, S_{2t}) \, dS_{1t} + \int_0^T \frac{\partial}{\partial S_2} g(S_{1t}, S_{2t}) \, dS_{2t}
\]

\[
\sim \frac{S_{1,t}}{v_{t}^2} + \frac{S_{2,t}}{v_{t}^2}
\]

\[
+ \int_0^T a_t^2 \left[ \frac{\sigma_1^2}{2} \frac{\partial^2}{\partial S_1^2} g(S_{1t}, S_{2t}) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial S_1 \partial S_2} g(S_{1t}, S_{2t}) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial S_2^2} g(S_{1t}, S_{2t}) ight]
\]

\[-V_I \delta(S_{1t} - K_1, S_{2t} - K_2)\]

Ito’s formula can be used on a truncated version of \( g \) we call \( g^\varepsilon \).

- Idea: Let \( A = (\Sigma^t \Sigma)^{-1} \) be the inverse of coefficient matrix. Surround singularity \((K_1, K_2)\) by an ellipsoid of radius \( \varepsilon \), \( \mathcal{E}_K(\varepsilon) = \{ X = (x_1, x_2) : X^t AX = \varepsilon \} \)

- Glue the solution \( g \) to the solution outside the ball to the solution inside the ball using a quadratic function, while matching both first and second derivatives. Glued together function is called \( g_\varepsilon \)
Therefore we obtain

\[
\frac{V_I}{4\pi \sqrt{\Delta \epsilon^2}} \int_0^T a_t^2 1_{(S_{1t},S_{2t}) \in \mathcal{E}_{\epsilon}^K} dt = g^\epsilon(S_{10},S_{20}) - g^\epsilon(S_{1T},S_{2T})
\]

from approximation to delta function

\[
+ \int_0^T \frac{\partial}{\partial S_1} g^\epsilon(S_{1t},S_{2t}) dS_{1t} + \int_0^T \frac{\partial}{\partial S_2} g^\epsilon(S_{1t},S_{2t}) dS_{2t}.
\]

where \( V_I = \sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 \),

- Differentiate this result with respect to \( T \)

\[
\frac{V_I}{4\pi \sqrt{\Delta \epsilon^2}} a_T^2 1_{(S_{1T},S_{2T}) \in \mathcal{E}_{\epsilon}^K} = -\frac{\partial}{\partial T} g^\epsilon(S_{1T},S_{2T}) + \frac{\partial}{\partial S_1} g^\epsilon(S_{1T},S_{2T}) dS_{1T} + \frac{\partial}{\partial S_2} g^\epsilon(S_{2T},S_{2T}) dS_{2T}
\]

- The left hand side is a payoff that captures an average stochastic volatility in an \( \epsilon \) neighborhood of the strike \( K = (K_1, K_2) \).
Normal Dynamics

• (from last slide)

\[
\frac{V_I}{4\pi \sqrt{\Delta \epsilon^2}} d^2 \mathbb{1}_{(S_{1t}, S_{2t}) \in \mathcal{E}_\epsilon^K} \\
= - \frac{\partial}{\partial T} g^\epsilon(S_{1T}, S_{2T}) + g^\epsilon(S_{1T}, S_{2T}) dS_{1T} + \frac{\partial}{\partial S_2} g^\epsilon(S_{2t}, S_{2t}) dS_{2t}
\]

• This payoff can be replicated by shorting a calendar spread of contingent \( \epsilon \) contingent claims with maturities \( T \) and \( T + \Delta T \) and by going long \( \frac{\partial g^\epsilon(S_{1T}, S_{2T})}{\partial S_1} \) shares of asset 1 and \( \frac{\partial g^\epsilon(S_{1T}, S_{2T})}{\partial S_2} \) shares of asset 2 during the time interval \([T, T + \Delta T]\)/

• Note the crucial role of the \( \epsilon \) approximating payoff. The fundamental solution being unbounded for \( n \geq 2 \), implies that the untruncated claim would not be a marketable claim.
Once again, as in 1-asset case, take risk-neutral expectations after having differentiated the result with respect to $T$ and obtain

$$\frac{V_I}{4\pi \sqrt{\Delta \epsilon^2}} EQ \left[ a_T^2 1_{(S_{1t}, S_{2t}) \in \mathcal{E}_\epsilon^K} \right] = -\frac{\partial}{\partial T} E_0 [g_T^\epsilon]$$

So that, with the same reasoning as before we obtain

$$E \left[ a_T^2 | (S_{1t}, S_{2t}) \in \mathcal{E}_\epsilon^K \right] = Q \left( (S_{1t}, S_{2t}) \in \mathcal{E}_\epsilon^K \right) \left( -\frac{\partial}{\partial T} E [-g_T^\epsilon] \right)$$

So if we know the risk-neutral probability that $(S_{1t}, S_{2t}) \in \mathcal{E}_\epsilon^K$ we have again the result that the averaged local variance can be captured by a calendar spread involving our $\epsilon$-payoff function $g^\epsilon(S, t)$. 
Modifications needed for Lognormal Dynamics

\{(S_{1t}, S_{2t}) : t \in [0, T]\}:

\[
dS_{1t} = S_{1t} \sigma_1 a_t dW_{1t}
\]

\[
dS_{2t} = S_{2t} \sigma_2 a_t dW_{2t}, \quad t \in [0, T],
\]

and initial spot prices \(S_{10} > 0\) and \(S_{20} > 0\). Here \(\sigma_1\) and \(\sigma_2\) are constants (this assumption will be relaxed later) and \(\{W_{1t} : t \in [0, T]\}\) and \(\{W_{2t} : t \in [0, T]\}\) are correlated one-dimensional standard Brownian motions, with constant correlation coefficient \(\rho\) i.e.:

\[
< W_{1t}, W_{2t} >_t = \rho t.
\]

Letting \(x_{it} = \ln(S_{it}), i = 1, 2\), and applying Itô’s formula, we may write the risk-neutral dynamics in the \(x\) variables as:

\[
dx_{it} = -\frac{\sigma_i^2}{2} a_i^2 dt + \sigma_i a_t dW_{it}, \quad t \in [0, T], i = 1, 2.
\]
Consider a function $U(x_1, x_2)$ which solves the following canonical version of the two dimensional constant coefficient elliptic equation:

$$\frac{\sigma_1^2}{2} \frac{\partial^2}{\partial x_1^2} U(x_1, x_2) + \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial x_1 \partial x_2} U(x_1, x_2) + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial x_2^2} U(x_1, x_2) + \nu_1 \frac{\partial}{\partial x_1} U(x_1, x_2) + \nu_2 \frac{\partial}{\partial x_2} U(x_1, x_2) = -V_I^\text{atm} \delta(x_1 - k_1, x_2 - k_2),$$

where $k_i \equiv \ln K_i, i = 1, 2$. A fundamental solution can be found in explicit form:

$$g(x, y, k) \equiv \frac{V_I}{\pi \sqrt{\triangle}} \exp \left[ -\frac{1}{2} \tilde{\nu}^t A (x - k) \right] K_0 \left[ Q \sqrt{(x - k)^t A (x - k)} \right],$$

where $\triangle$ again denotes the determinant of the covariance matrix and $\tilde{\nu}$ is the vector of drifts:

$$\tilde{\nu} = [2\nu_1, 2\nu_2] = [-\sigma_1^2, -\sigma_2^2]$$

the matrix $A \equiv a^{-1}$ is the inverse of $a$, $K_0$ is the modified Bessel function of the second kind and degree zero, and $Q$ is the constant $\tilde{\nu}^t A \tilde{\nu}$. 
Higher dimension: Lognormal dynamics

Constant coefficients: Everything generalizes naturally.
Let us consider again the stochastic differential equations:

\[ dS_{it} = a_t \sigma_{ij} dW_{ijt}, \quad i = 1, \ldots, n, \]

where now \( \sigma_{ij} = \sigma_{ij}(S_1, \ldots, S_n) \) can depend on the stock prices \( S \), but not on time. With \( a_{ij} = \text{Trace}(\sigma \sigma^t) \), and \( x = \ln(S) \).

We again are lead to consider the following variable coefficient elliptic equation:

\[ Lu = a_{ij}(x_1, \ldots, x_n) u_{x_i x_j} + \nu_i(x_1, \ldots, x_n) u_{x_i} = -\delta(x - k), \quad (10) \]

in the log variables \( x_i = \ln S_i, k_i = \ln K_i, i = 1, \ldots, n \). Here, as before, \( \nu_i = -\sum_{i=1}^{n} \frac{\sigma_{ij}^2}{2} \).

In order to explain Hadamard's approach, we need to recall the idea of geodesic distance.
0.1 Riemannian Metrics and Geodesic Distance

Consider the inverse $A = a^{-1}$ of the coefficient matrix $a$. A Riemannian metric is defined by:

$$ds^2 = A_{ij} dx_i dx_j.$$  \hfill (11)

The geodesics are curves $x(\tau) = (x_1(\tau), \ldots, x_n(\tau))$ that minimize the integral:

$$I = \int_{\tau_0}^{\tau_1} \sqrt{\sum_{i,j=1}^{n} A_{ij} \dot{x}_i \dot{x}_j} d\tau$$

between fixed limits:

$$x(\tau_0) = \hat{x}, x(\tau_1) = x.$$ 

Consider the square $\Gamma = I^2$ of the geodesic distance as a function of its upper limit $x$. It can be shown that this function solves the following first order partial differential equation:

$$a_{ij} x_i \Gamma x_j = 4\Gamma.$$  \hfill (12)
Setting $p_i = \Gamma_{x_i}, i, \ldots, n$, the Hamiltonian associated to this PDE is:

$$H = \frac{1}{4} a_{ij} p_i p_j,$$

geodesics are determined by integrating the system of so-called ray equations:

$$\frac{dx_i}{dt} = \frac{1}{2} a_{ij} p_j, \quad \frac{dp_i}{ds} = -\frac{1}{4} \frac{\partial a_{jk}}{\partial x_i} p_i p_k, \quad i = 1, \ldots, n,$$

subject to the initial conditions:

$$x_i(0) = \hat{x}_i, p_i(0) = q_i.$$

The $q_i$ parametrize the direction of the tangent line to the geodesic through the base point $\hat{x}$ in the tangent plane to the surface defined by the Riemannian surface associated to our metric $A_{ij} ds_i ds_j$. 
Hadamard’s Method for Determining the Fundamental Solution

We wish to solve

$$a_{ij}(x, t)u_{x_i x_j} + b_i(x, t)u_{x_i} = -\delta(x - K),$$

i.e. to determine a fundamental solution of the elliptic equation with variable coefficients.

There are several methods.

Perhaps the most geometrically appealing and most constructive in flavour is Hadamard’s method.

(Brief Introduction to Hadamard’s Method)

Recall the Riemannian metric associated to the principal part of the elliptic differential operator. I.e.

$$A_{ij} ds_i ds_j$$

where

$$a = a^{-1} \quad a = \{a_{ij}\}$$
There are some technical differences in the method according as to whether the underlying spatial dimension is even or odd. For reasons of brevity, here we limit our description to the case of an odd dimension.

( Odd \( n \geq 3 \) )

\[
F = \frac{U}{\Gamma^{\frac{n-2}{2}}},
\]

(13)

where \( m \equiv \frac{n-2}{2} \) and where \( U \) is an infinite series of the following form:

\[
U_0 + U_1 \Gamma + U_2 \Gamma^2 + \ldots = \sum_{l=1}^{+\infty} U_l \Gamma^l.
\]

where \( \Gamma \) is the square of the geodesic distance in the Riemannian metric defined above.
System of ODE’s along geodesics

Plugging this ansatz into the PDE and collecting terms multiplying the same power of $\Gamma$ leads to the iterative determination of solution by solving system of ODE’s

We have:

\[ s \frac{dU_0}{ds} + \frac{M - 2n}{4} U_0 = 0 \]

\[ 4s \frac{dU_j}{ds} + (M + 4j - 2n)U_j - \frac{LU_{j-1}}{j - m} = 0, \quad j \geq 1. \]

By means of the substitution $y = \ln s$ and using an integrating factor, the second set of equations can be written as:

\[ \frac{d}{ds} \left( \frac{s^j dU_j}{U_0} \right) = \frac{s^{j-1} LU_{j-1}}{4(j - m) U_0}, \quad l \geq 1, \]

where here: \[ M \equiv -L(\Gamma). \]
The solution of these equations can be iteratively determined in the form:

\[
U_0(x, x') = \frac{\gamma(m)}{4\pi \frac{n}{2}} \exp \left( - \int_0^s \frac{M(x(u), y) - 2n}{4u} du \right)
\]

\[
U_j(x, y) = \frac{U_0(x, y)}{4(j - m)s^j} \int_0^s \frac{u^{j-1}(LU_{j-1})(x(u), y)}{U_0(x, y)} du, \quad j \geq 1.
\]

Here, \(x(u)\) denotes the path on the unique geodesic joining \(y\) and \(x\).

Hadamard shows that for small enough neighborhood of center the resulting series converges, provided the coefficients of the solution are assumed to be analytic.

The geodesic neighborhood must also be taken to be sufficiently small so as to ensure that geodesics do not cross.
Idea of Construction of $g^\epsilon$ payoff in the simplest

- We will illustrate the construction in the simplest $2 - D$ case where the stocks follow a normal dynamics.
- Idea of construction is a natural extension of a similar construction used in one of the proofs of the Meyer-Tanaka formula for local time.
- (Tanaka) Generalized Ito’s formula for convex functions with corner. For illustration consider simpler Tanaka’s formula for taking stochastic differential of $g(W_t)$ where

$$f(x) = |x|$$

It says (we saw it at the beginning of the talk):

$$|W_t| = |W_0| + \int_0^t \text{sign}(W_s)dW_s + L_t(\omega)$$

where $L_t$, the local time at zero defined by

$$L_t = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} |\{s \in [0, t]; W_s \in (-\epsilon, \epsilon)\}| \quad \text{limit in } L^2(P) \text{ sense}$$
One possible proof of Tanaka formula approximates $|x|$ by a quadratic function in the region $|x| \leq \varepsilon$.

$$g^\varepsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \varepsilon \\ \frac{1}{2}(\varepsilon + \frac{x^2}{2}) & \text{if } |x| < \varepsilon \end{cases}$$
Illustration Tanaka formula

The figure shows a parabola in the region where the function $f(x) = |x|$ is defined.
Approximating Tanaka formula

- \( g^\epsilon(x) \) approximating function we obtain the approximating formula

\[
g^\epsilon(W_t) = g^\epsilon(W_0) + \int_0^t (g^\epsilon)'(W_s)dW_s + \frac{1}{2\epsilon} \left| \{s \in [0, t]; W_s \in (-\epsilon, \epsilon)\} \right|
\]

- What is the analogue to this approach in dimension \( n \geq 2 \)?
  Answer: Replace the fundamental solution inside the ellipsoid by a multivariate quadratic function in such a way that:
  i) The function and its first derivatives paste smoothly along the boundary of the ellipsoid
  ii) The first derivatives are piecewise \( C^1 \) and the jumps in the second derivatives occur but these are globally bounded.

- For \( n = 2 \) such a quadratic function can be shown to be

\[
\bar{g}(x, \xi) = -\frac{1}{4\pi \sqrt{\Delta} \epsilon^2} A_2(x, \xi) + \frac{1}{4\pi \sqrt{\Delta}} (1 - \ln(\epsilon^2)) \quad \text{for } n = 2
\]

where \( A_2(x, \xi) = -\sum_{i, j=1}^2 A_{ij}(x_i - \xi_i) \) (recall \( A = a^{-1} \)).
Radon-Transform Approach to multi-D Breeden

Payoff of a basket option:

\[(\sum w_i S_i - K)^+\]

Normalize this payoff

\[|w|(\sum w_i' S_i - q)^+\]

where \(\xi_i = \frac{w_i}{|w|}, \quad q = \frac{K}{|w|}\)

Consider \(C(\xi, q)\) i.e. the prices of all basket options with \(\xi \in S^{n-1}, q \in [0, \infty]\). Let \(p(S_1, \cdots, S_n)\) denote the risk-neutral density, so that

\[C(\xi, q) = \int_{\mathbb{R}^+} (\xi \cdot S - q)^+ p(S_1, \cdots, S_n) dS_1 \cdots dS_n\]
\[
\frac{\partial^2}{\partial q^2} C(\xi, q) = \int_{S \cdot \xi - q = 0} p(S_1, \cdots, S_n) dS_1 \cdots dS_n
\]

ie. \( C(\xi, q) \) corresponds to the integral of density \( p \) on a hyperplane with normal \( \xi \in S^{n-1} \) and a distance \( q \) from the origin.

\[
\frac{\partial^2}{\partial q^2} C(\xi, q) = \mathcal{R}[p](\xi, q)
\]

where \( \mathcal{R}[p](\xi, q) \) is the Radon transform of the density \( p \).
Now key observation is that there is an inversion formula for the Radon transform which reads

\[
v(\xi, q) = \mathcal{R}[p](\xi, q)
\]

\[
\iff
\]

\[
p(S_1, \cdots S_n) = \int_{|\xi|=1} h(S \cdot \xi, \xi) d\xi
\]

Here definition of \( h \) depends on whether dimension is \( n \) is even or odd:

\[
h(q, \xi) \equiv \frac{(i)^{n-1}}{2(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial q^{n-1}} R[f](\xi, q) \quad n \text{ odd}
\]

\[
h(q, \xi) \equiv \frac{(i)^n}{2(2\pi)^{n-1}} \mathcal{H} \left[ \frac{\partial^{n-1}}{\partial p^{n-1}} R[f](\xi, q) \right] (q) \quad n \text{ even}
\]

where \( \mathcal{H}[g(p)](t) \) denotes the Hilbert transform of the function \( g(p) \):

\[
\mathcal{H}[g(p)](t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(p)}{p - t} dp,
\]
Closed form solutions for option pricing and interest models on two assets
The result is to appear in

Black-Scholes equation

\[ \frac{\partial C}{\partial \tau} + \frac{\sigma^2}{2} S^2 \frac{\partial C^2}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \]
Black-Scholes equation

\[
\frac{\partial C}{\partial \tau} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0
\]

\[
\frac{\partial C}{\partial t} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C
\]

\[ t = T - \tau \]
Black-Scholes equation

\[
\frac{\partial C}{\partial \tau} + \frac{\sigma^2}{2} S^2 \frac{\partial C^2}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0
\]

\[t = T - \tau\]

\[
\frac{\partial C}{\partial t} = \frac{\sigma^2}{2} S^2 \frac{\partial C^2}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC
\]

\[\xi = \ln S\]

\[
\frac{\partial C}{\partial t} = \frac{\sigma^2}{2} \frac{\partial C^2}{\partial \xi^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial C}{\partial \xi} - rC
\]
Black-Scholes equation

\[ c(\xi, t) = e^{rt} C(\xi, t) \]

\[
\frac{\partial c}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial \xi^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial \xi}
\]
Black-Scholes equation

\[ c(\xi, t) = e^{rt} C(\xi, t) \]

\[
\frac{\partial c}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial \xi^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial \xi} + \frac{\mu}{2} \left( 1 + \frac{\sigma^2}{2} \right) t
\]

\[ x = \xi + \left( r - \frac{\sigma^2}{2} \right) t \]

\[
\frac{\partial c}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial x^2}
\]
In total, we have done the transformation

\[ t = T - \tau \]
\[ x = \ln S + \left( r - \frac{\sigma^2}{2} \right) (T - \tau) \]
\[ c = e^{r(T-\tau)} C' \]

which transforms Black-Scholes equation to heat equation.
Merton’s arguments in his 1973 paper imply more generally that the arbitrage-free value $C$ of many derivatives satisfies

$$\frac{\partial C}{\partial \tau} + \frac{\sigma^2(S, \tau)}{2} S^2 \frac{\partial C^2}{\partial S^2} + b(S, \tau) \frac{\partial C}{\partial S} - r(S, \tau)C = 0$$

with three variable coefficients $\sigma(S, \tau)$, $b(S, \tau)$ and $r(S, \tau)$. 
Merton’s arguments in his 1973 paper imply more generally that the arbitrage-free value \( C \) of many derivatives satisfies

\[
\frac{\partial C}{\partial \tau} + \frac{\sigma^2(S, \tau)}{2} S^2 \frac{\partial C^2}{\partial S^2} + b(S, \tau) \frac{\partial C}{\partial S} - r(S, \tau)C = 0
\]

with three variable coefficients \( \sigma(S, \tau), b(S, \tau) \) and \( r(S, \tau) \).

Carr-Lipton-Madan(2000) characterized the entire set of variable coefficients \( \{\sigma(S, \tau), b(S, \tau), r(S, \tau)\} \) which permit the above one state variable valuation PDE to be transformed to the heat equation.
Lie’s group classification of linear second order PDE with 2 independent variables

\[ Pu_t + Qu_x + Ru_{xx} + Su = 0 \]

\( P, R \neq 0 \) and \( P, Q, R, S \) are functions of \( t \) and \( x \).

- Any equation is admitted by the generators of the trivial symmetries \( u \frac{\partial}{\partial u} \) and \( \phi(t, x) \frac{\partial}{\partial u} \). Any equation can be reduced to the form

\[ v_\tau = v_{yy} + Z(\tau, y)v \]

by a transformation as \( y = \alpha(t, x), \tau = \beta(t) \) and \( v = \gamma(t, x)u \) with \( \alpha_x \neq 0 \) and \( \beta_t \neq 0 \).
An equation admits symmetry generators generated by

\[ u \frac{\partial}{\partial u}, \quad \phi(t, x) \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \tau} \]

can be reduced to the form

\[ \nu_\tau = \nu_{yy} + Z(y) \nu \]

by a transformation.
An equation admits symmetry generators generated by

\[ u \frac{\partial}{\partial u}, \quad \phi(t, x) \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \tau}, \quad 2\tau \frac{\partial}{\partial \tau} + y \frac{\partial}{\partial y} \]

\[ \tau^2 \frac{\partial}{\partial \tau} + \tau y \frac{\partial}{\partial y} - \left( \frac{1}{4} y^2 + \frac{1}{2} \tau \right) v \frac{\partial}{\partial v} \]

can be reduced to the form

\[ v_\tau = v_{yy} + \frac{A}{y^2} v \]

by a transformation, where \( A \) is a constant.
An equation admits symmetry generators generated by

\[
\begin{align*}
    u \frac{\partial}{\partial u}, \quad \phi(t, x) \frac{\partial}{\partial u}, \quad & \frac{\partial}{\partial \tau} \quad 2\tau \frac{\partial}{\partial \tau} + y \frac{\partial}{\partial y}, \\
    \tau^2 \frac{\partial}{\partial \tau} + \tau y \frac{\partial}{\partial y} - \left( \frac{1}{4} y^2 + \frac{1}{2} \tau \right) v \frac{\partial}{\partial v}, \\
    \frac{\partial}{\partial y}, \quad & 2\tau \frac{\partial}{\partial y} - yv \frac{\partial}{\partial v}
\end{align*}
\]

can be reduced to the form

\[ v_\tau = v_{yy} \]

by a transformation.
Convenient Variables

The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied.

- Jacobi, *Lectures on Dynamics*, 1847.
From geometric point of view, the great Norwegian mathematician Sophus Lie created a systematic way for analyzing solutions of differential equations. His theory nowadays has been known as Lie’s symmetry analysis of differential equation. Up to now, Lie’s theory has been greatly extended and used to find closed form solutions, to classify equations up to some unknown functions, to find fundamental solutions and so on.
A symmetry group of a system $S$ of differential equations is a local group of transformations, $G$, acting on some open subset $M \subset X \times U$ in such a way that "$G$ transforms solutions of $S$ to other solutions of $S$".

- Allowing arbitrary nonlinear transformations of both the independent and dependent variables in the definition of symmetry.
Prolongation of $f$

$f : X \to U$, $u = f(x)$ $u^{(n)} := \text{pr}^{(n)} f(x) : X \to U^{(n)}$: the $n$-th prolongation of $f$ is defined by

$$u_J = \partial_J f(x)$$

For example, $u = f(x, y)$, the second prolongation $u^{(2)} = \text{pr}^{(2)} f(x, y)$ is given by

$$(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = \left( f; \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2} \right).$$
Solutions of DE’s

- An $n$-th order differential equation in $p$ independent variable is given as

$$\Delta(x, u^{(n)}) = 0$$

involving $x = (x^1, \cdots, x^p)$, $u$ and the derivatives of $u$ with respect to $x$ up to order $n$.

- $\Delta$ can be viewed as a function from the jet space $X \times U^{(n)}$ to $\mathbb{R}$, i.e., $\Delta : X \times U^{(n)} \to \mathbb{R}$.

- Hence, an $n$-th order differential equation can be regarded as a subvariety $S_\Delta := \{\Delta(x, u^{(n)}) = 0\}$ in the $n$-jet space $X \times U^{(n)}$. 
A smooth solution of the given $n$-th order differential equation in this sense is a smooth function $u = f(x)$ such that

$$\Delta(x, \text{pr}^{(n)} f(x)) = 0$$

whenever $x$ lies in the domain of $f$.

Hence a solution can be regarded as a function $u = f(x)$ such that the graph of the $n$-th prolongation $\text{pr}^{(n)} f$ is contained in the subvariety $S_\Delta$, i.e.,

$$\Gamma_f^{(n)} := \{(x, \text{pr}^{(n)} f(x))\} \subset S_\Delta$$
Prolongation of group actions

- $G$: local group of transformations acting on $M \subset X \times U$

- The $n$-th prolongation $\text{pr}^{(n)}G$ of $G$ is an induced local action of $G$ on the $n$-jet space $M^{(n)}$ which is defined so that it transforms the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$.

- This can be done by calculating the transformation of $n$-th order Taylor polynomial.
Invariance of DEs

\[ \Delta(x, u^{(n)}) = 0: \text{n-th order differential equation defined over } M \]

\[ S_\Delta \subset M^{(n)}: \text{corresponding subvariety} \]

\[ G: \text{local group of transformations acting on } M \text{ whose prolongation leaves } S_\Delta \text{ invariant, i.e.,} \]

\[ (x, u^{(n)}) \in S_\Delta \Rightarrow \text{pr}^{(n)}_g \cdot (x, u^{(n)}) \in S_\Delta \text{ for all } g \in G. \]
Invariance of DEs

- $\Delta(x, u^{(n)}) = 0$: $n$-th order differential equation defined over $M$

- $S_\Delta \subset M^{(n)}$: corresponding subvariety

- $G$: local group of transformations acting on $M$ whose prolongation leaves $S_\Delta$ invariant, i.e.,

$$ (x, u^{(n)}) \in S_\Delta \Rightarrow \text{pr}^{(n)}(g \cdot (x, u^{(n)})) \in S_\Delta \text{ for all } g \in G. $$

$G$ is a symmetry group of the differential equation!
The $n$-th prolongation $\text{pr}^{(n)}v$ of $v$ is a vector field on the jet space $M^{(n)}$ and defined to be the infinitesimal generator of the corresponding prolonged group $\text{pr}^{(n)}[\exp(\epsilon v)]$, i.e.,

$$\text{pr}^{(n)}v|_{(x,u^{(n)})} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{pr}^{(n)}[\exp(\epsilon v)](x,u^{(n)})$$

for any $(x,u^{(n)}) \in M^{(n)}$. 

- $v$: vector field on $M$
- $\exp(\epsilon v)$: corresponding one-parameter group
Infiniteesimal invariance

**Lie’s Theorem**

\[ \Delta(x, u^{(n)}) : n\text{-th order differential equation} \]

- Let \( \mathbf{v} \) be a vector field. Then \( \mathbf{v} \) generates a one parameter local group of symmetries if and only if

\[
\text{pr}^{(n)} \mathbf{v} [\Delta(x, u^{(n)})] = 0
\]

whenever \( \Delta(x, u^{(n)}) = 0 \).

- The set of all infiniteesimal symmetries form a Lie algebra of vector fields on \( X \times U \). Moreover, if this Lie algebra is finite dimensional, the symmetry group of the equation is a local Lie group of transformations acting on \( X \times U \).
The second prolongation $\text{pr}^{(2)} v$ of a vector field $v$ with

$$v = \sum_i \xi^i(t, x, u) \frac{\partial}{\partial x^i} + \phi(t, x, u) \frac{\partial}{\partial u}$$

is as

$$\text{pr}^{(2)} v = v + \sum_i \phi^i \frac{\partial}{\partial u x^i} + \sum_{i,j} \phi^{ij} \frac{\partial}{\partial u x^i x^j}$$

where the $\phi$’s are given by the prolongation formulas
Total differentiation

\[ \phi^i = D_i(\phi) - \sum_j u_{x^j} D_i(\xi^j) \]

\[ \phi^{ij} = D_j(\phi^i) - \sum_k u_{x^i x^k} D_j(\xi^k) \]

and the \( D \)'s are the so-called total differentiation.

\[ D_i = \frac{\partial}{\partial x^i} + u_{x^i} \frac{\partial}{\partial u} + u_{x^i x^k} \frac{\partial}{\partial u_{x^k}} + \cdots \]
The determining equation

\[ \text{pr}^{(n)} v[\Delta(x, u^{(n)})] = 0 \text{ whenever } \Delta(x, u^{(n)}) = 0 \]

is a linear homogeneous partial differential equation of the second order for the unknown functions \( \xi, \tau, \phi \) of the "independent variables" \( t, x, u \).
The determining equation

\[ p_{r}^{(n)} v[\Delta(x, u^{(n)})] = 0 \text{ whenever } \Delta(x, u^{(n)}) = 0 \]

is a linear homogeneous partial differential equation of the second order for the unknown functions \( \xi, \tau, \phi \) of the "independent variables" \( t, x, u \).

- \( t, x, u, u_{x}, u_{x_{i}}, u_{x_{i}x_{j}}, \cdots \), are regarded as "independent" ones.
The determining equation

\[ \text{pr}^{(n)} v[\Delta(x, u^{(n)})] = 0 \text{ whenever } \Delta(x, u^{(n)}) = 0 \]

is a linear homogeneous partial differential equation of the second order for the unknown functions \( \xi, \tau, \phi \) of the "independent variables" \( t, x, u \).

- \( t, x, u, u_{x^i}, u_{x^i x^j}, \ldots \), are regarded as "independent" ones.
- The determining equation decomposes into a system of several equations.
The determining equation

\[ p^{(n)}_r v[\Delta(x, u^{(n)})] = 0 \text{ whenever } \Delta(x, u^{(n)}) = 0 \]

is a linear homogeneous partial differential equation of the second order for the unknown functions \( \xi, \tau, \phi \) of the "independent variables" \( t, x, u \).

- \( t, x, u, u_{x^i}, u_{x^i x^j}, \ldots \), are regarded as "independent" ones.
- The determining equation decomposes into a system of several equations.
- The resulting system is an overdetermined system which can be solved analytically.
1 dim heat equation

- $u_t = u_{xx}$

- The heat equation can be identified with the linear subvariety in $X \times U^{(2)}$ determined by the vanishing of
  $\Delta(x, t, u^{(2)}) = u_t - u_{xx}$.

- $v = \xi(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \phi(t, x, u) \frac{\partial}{\partial u}$

- $\text{pr}^{(2)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}$
Infinitesimal invariance

- \[ \text{pr}^{(2)} v[\Delta(x, t, u^{(2)})] = \phi^t - \phi^{xx} = 0 \]
- \[ \phi^t = \phi_t - \xi_t u_x + (\phi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u^2_t \]
- \[ \phi^{xx} = \phi_{xx} + (2\phi_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t + \xi_{uu} u^3_x - \tau_{uu} u^2_x u_t + (\phi_u - 2\xi_x) u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt} \]
- Replacing \( u_t \) by \( u_{xx} \)
### Monomial

<table>
<thead>
<tr>
<th>Term</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_x u_{xt}$</td>
<td>$-2\tau_u = 0$</td>
</tr>
<tr>
<td>$u_{xt}$</td>
<td>$-2\tau_x = 0$</td>
</tr>
<tr>
<td>$u_{xx}^2$</td>
<td>$-\tau_u = -\tau_u$</td>
</tr>
<tr>
<td>$u_x u_{xx}$</td>
<td>$0 = -\tau_{uu}$</td>
</tr>
<tr>
<td>$u_x u_{xxx}$</td>
<td>$-\xi_u = -2\tau_{xu} - 3\xi_u$</td>
</tr>
<tr>
<td>$u_{xxx}$</td>
<td>$\phi_u - \tau_t = -\tau_{xx} + \phi_u - 2\xi_x$</td>
</tr>
<tr>
<td>$u_x^3$</td>
<td>$0 = -\xi_{uu}$</td>
</tr>
<tr>
<td>$u_x^2$</td>
<td>$0 = \phi_{uu} - 2\xi_{xu}$</td>
</tr>
<tr>
<td>$u_x$</td>
<td>$-\xi_t = 2\phi_{xu} - \xi_{xx}$</td>
</tr>
<tr>
<td>1</td>
<td>$\phi_t = \phi_{xx}$</td>
</tr>
</tbody>
</table>
General solution

- \( \xi(x, t) = c_1 + c_4x + 2c_5t + 4c_6xt \)
- \( \tau(t) = c_2 + 2c_4t + 4c_6t^2 \)
- \( \phi(x, t, u) = (c_3 - c_5x - 2c_6t - c_6x^2)u + \alpha(x, t) \)
- \( c_1, \cdots, c_6 \) are arbitrary constants and \( \alpha(x, t) \) an arbitrary solution of the heat equation.
Finite (six) dimensional subalgebra

\[ \mathbf{v}_1 = \partial_x \]
\[ \mathbf{v}_2 = \partial_t \]
\[ \mathbf{v}_3 = u \partial_u \]
\[ \mathbf{v}_4 = x \partial_x + 2t \partial_t \]
\[ \mathbf{v}_5 = 2t \partial_x - xu \partial_u \]
\[ \mathbf{v}_6 = 4tx \partial_x + 4t^2 \partial_t - (x^2 + 2t)u \partial_u \]

Infinite dimensional subalgebra

\[ \mathbf{v}_\alpha = \alpha(x, t) \partial_u, \]

where \( \alpha \) is an arbitrary solution of the heat equation.
Symmetry group

\(G_1: \ (x + \epsilon, t, u)\)

\(G_2: \ (x, t + \epsilon, u)\)

\(G_3: \ (x, t, e^\epsilon u)\)

\(G_4: \ (e^\epsilon x, e^{2\epsilon} t, u)\)

\(G_5: \ (x + 2\epsilon t, t, u \cdot \exp(-\epsilon x - \epsilon^2 t))\)

\(G_6: \ \left(\frac{x}{1 - 4\epsilon t}, \frac{t}{1 - 4\epsilon t}, u \sqrt{1 - 4\epsilon t e^{\frac{-\epsilon x^2}{1 - 4\epsilon t}}}\right)\)

\(G_\alpha: \ (x, t, u + \epsilon\alpha(x, t))\)
Symmetry group

- $G_3, G_\alpha$: the linearity of the heat equation

- $G_1, G_2$: time and space invariance of the equation, reflecting that heat equation is of constant coefficients.

- $G_4$: well-known scaling symmetry

- $G_5$: Galilean boost to a moving coordinate frame

- $G_6$: fundamental solution
If $u = f(x, t)$ is a solution of the heat equation, so are the following functions

$$
\begin{align*}
    u^1 &= f(x - \epsilon, t) \\
    u^2 &= f(x, t - \epsilon) \\
    u^3 &= e^\epsilon f(x, t) \\
    u^4 &= f(e^{-\epsilon}x, e^{-2\epsilon}t) \\
    u^5 &= e^{-\epsilon x + \epsilon^2 t} f(x - 2\epsilon t, t) \\
    u^6 &= \frac{1}{\sqrt{1 + 4\epsilon t}} e^{-\epsilon^2 x^2} \left[ e^{1+4\epsilon t} f\left(\frac{x}{1 + 4\epsilon t}, \frac{t}{1 + 4\epsilon t}\right) \right] \\
    u^\alpha &= f(x, t) + \epsilon \alpha(x, t)
\end{align*}
$$
Finkel has classified the 2 dimensional parabolic equations of the form

\[ u_t - \frac{1}{2} \Delta u + M(x, y)u = 0. \]

Symmetry operators in the following are of the form

\[ X = \tau(t) \partial_t + \xi(x, y, t) \partial_x + \eta(x, y, t) \partial_y + \varphi(x, y, t) u \partial_u \]

where

\[ \xi = \frac{\tau_t}{2} + \gamma y + \xi_0(t) \]
\[ \eta = \frac{\tau_t}{2} - \gamma x + \eta_0(t) \]
\[ \varphi = -\frac{\tau_{tt}}{4} (x^2 + y^2) - \xi'_0 x - \eta'_0 y + \phi(t) \]
Case 1.1a, Dimension of group = 4

\[ M = \frac{C_0}{x^2} + by + c_0, \quad C_0 \neq 0 \]

\[ \tau = \delta_2 t^2 + \delta_1 t, \quad \gamma = \xi_0 = 0, \quad \eta_0 = \frac{b\delta_2}{2} t^3 + \frac{3b\delta_1}{4} t^2 + \beta_1 t + \beta_0, \]

\[ \phi = -\frac{b^2\delta_2}{8} t^4 - \frac{b^2\delta_1}{4} t^3 - \left( \frac{b\beta_1}{2} + c_0\delta_2 \right) t^2 - (\delta_2 + c_0\delta_1 + b\beta_0) t \]
Case 1.1b, Dimension of group = 4

\[ M = \frac{C_0}{x^2} + cr^2 + by + c_0, \quad C_0 \neq 0 \]

\[ \tau = \delta_1 e^{2\sqrt{2ct}} + \delta_2 e^{-2\sqrt{2ct}}, \quad \gamma = \xi_0 = 0, \]

\[ \eta_0 = \frac{b\delta_1}{\sqrt{2c}} e^{2\sqrt{2ct}} - \frac{b\delta_2}{\sqrt{2c}} e^{-2\sqrt{2ct}} + \beta_1 e^{\sqrt{2ct}} + \beta_2 e^{-\sqrt{2ct}}, \]

\[ \phi = -\left( \sqrt{2c} + c_0 + \frac{b^2}{4c} \right) \delta_1 e^{2\sqrt{2ct}} + \left( \sqrt{2c} - c_0 - \frac{b^2}{4c} \right) \delta_2 e^{-2\sqrt{2ct}} \]

\[ -\frac{b\beta_1}{\sqrt{2c}} e^{\sqrt{2ct}} + \frac{b\beta_2}{\sqrt{2c}} e^{-\sqrt{2ct}} \]
Case 1.2a, Dimension of group = 2

\[ M = \frac{C(\theta)}{r^2} + c_0 \]

\[ \tau = \delta_2 t^2 + \delta_1 t, \quad \gamma = \xi_0 = \eta_0 = 0, \quad \phi = -c_0 \delta_2 t^2 - (\delta_2 + c_0 \delta_1) t, \]

where \( C(\theta) \neq (C_0 \cos \theta + C_1 \sin \theta)^{-2} \) and \( C' \neq 0 \).
Case 1.2b, Dimension of group = 2

\[ M = C'(\theta) \frac{1}{r^2} + cr^2 + c_0 \]

\[ \tau = \delta_1 e^{2\sqrt{2}ct} + \delta_2 e^{-2\sqrt{2}ct}, \quad \gamma = \xi_0 = \eta_0 = 0, \]

\[ \phi = -\left(\sqrt{2}c + c_0\right) \delta_1 e^{2\sqrt{2}ct} + \left(\sqrt{2}c - c_0\right) \delta_1 2e^{-2\sqrt{2}ct} \]

with \( C'(\theta) \neq (C_0 \cos \theta + C_1 \sin \theta)^{-2} \) and \( C' \neq 0 \).
Case 1.3, Dimension of group = 1

\[ M = C(\lambda \log r + \theta) \frac{1}{r^2} + c_0, \quad C' \neq 0 \neq \lambda \]

\[ \tau = \frac{2\gamma}{\lambda} t, \quad \xi_0 = \eta_0 = 0, \quad \phi = -\frac{2c_0\gamma}{\lambda} t \]
Case 1.4a, Dimension of group = 3

\[ M = C_0 \frac{1}{r^2} + ax + by + c_0, \quad C_0 \neq 0 \]

\[ \tau = \delta_2 t^2 + \delta_1 t, \quad \xi_0 = \eta_0 = 0, \quad \phi = -c_0 \delta_2 t^2 - (\delta_2 + c_0 \delta_1) t \]

with the constraints \( \delta_1 = \delta_2 = 0 \) if \( a \neq 0 \) or \( b \neq 0 \).
Case 1.4b, Dimension of group = 3

\[ M = C_0 r^{-2} + cr^2 + ax + by + c_0, \quad C_0 \neq 0 \]

\[ \tau = \delta_1 e^{2\sqrt{2ct}} + \delta_2 e^{-2\sqrt{2ct}b}, \quad \xi_0 = \eta_0 = 0, \]
\[ \phi = -(\sqrt{2c} + c_0)\delta_1 e^{2\sqrt{2ct}} + (\sqrt{2c} - c_0)\delta_2 e^{-2\sqrt{2ct}} \]

with the constraint \( \delta_1 = \delta_2 = 0 \) if \( a \neq 0 \) or \( b \neq 0 \).
Case 1.5a, Dimension of group = 7

\[ M = ax + by + c_0 \]

\[ \tau = \delta_2 t^2 + \delta_1 t, \]

\[ \xi_0 = \frac{a\delta_2}{2} t^3 + \frac{1}{4} (3a\delta_1 - 2b\gamma) t^2 + \alpha_1 t + \alpha_0, \]

\[ \eta_0 = \frac{b\delta_2}{2} t^3 + \frac{1}{4} (3b\delta_1 + 2a\gamma) t^2 + \beta_1 t + \beta_0, \]

\[ \phi = -\frac{1}{8} (a^2 + b^2) \delta_2 t^4 - \frac{1}{4} (a^2 + b^2) \delta_1 t^3 - \left( \frac{1}{2} (a\alpha_1 + b\beta_1) + c_0 \delta_2 \right) t^2 \]

\[-(\delta_2 + c_0 \delta_1 + a\alpha_0 + b\beta_0) t \]
Case 1.5b, Dimension of group = 7

\[ M = cr^2 + ax + by + c_0 \]

\[
\tau = \delta_1 e^{2\sqrt{2}ct} + \delta_2 e^{-2\sqrt{2}ct},
\]
\[
\xi_0 = \frac{a\delta_1}{\sqrt{2c}} e^{2\sqrt{2}ct} + \frac{a\delta_2}{\sqrt{2c}} e^{-2\sqrt{2}ct} + \alpha_1 e^{\sqrt{2}ct} + \alpha_2 e^{-\sqrt{2}ct} + \frac{b\gamma}{2c},
\]
\[
\eta_0 = \frac{b\delta_1}{\sqrt{2c}} e^{2\sqrt{2}ct} + \frac{b\delta_2}{\sqrt{2c}} e^{-2\sqrt{2}ct} + \beta_1 e^{\sqrt{2}ct} + \beta_2 e^{-\sqrt{2}ct} - \frac{a\gamma}{2c},
\]
\[
\phi = -\left( \sqrt{2c} + c_0 + \frac{a^2 + b^2}{4c} \right) \delta_1 e^{2\sqrt{2}ct} + \left( \sqrt{2c} - c_0 - \frac{a^2 + b^2}{4c} \right) \delta_2 e^{-2\sqrt{2}ct}
\]
\[-\frac{a\alpha_1 + b\beta_1}{\sqrt{2c}} e^{\sqrt{2}ct} + \frac{a\alpha_2 + b\beta_2}{\sqrt{2c}} e^{-\sqrt{2}ct} \]
Case 1.6, Dimension of group = 1

\[ M = C'(r) + d\theta \]

\[ \tau = \xi_0 = \eta_0 = 0, \quad \phi = d\gamma t \]

with \( C'(r) \neq C_0 r^{-2} + C_1 r^2 + c_0 \) if \( d = 0 \).
Case 1.7a, Dimension of group = 4

\[ M = cx^2 + ax + by + c_0 \]

\[ \tau = \gamma = 0, \quad \xi_0 = \alpha_1 e^{\sqrt{2}ct} + \alpha_2 e^{-\sqrt{2}ct}, \quad \eta_0 = \beta_1 t + \beta_0, \]

\[ \phi = -\frac{a\alpha_1}{\sqrt{2}c} e^{\sqrt{2}ct} + \frac{a\alpha_2}{\sqrt{2}c} e^{-\sqrt{2}ct} - \frac{b\beta_1}{2} t^2 - b\beta_0 t \]
Case 1.7b, Dimension of group = 4

\[ M = c(x^2 - y^2) + ax + by + c_0 \]

\[ \tau = \gamma = 0, \quad \xi_0 = \alpha_1 e^{\sqrt{2}ct} + \alpha_2 e^{-\sqrt{2}ct}, \quad \eta_0 = \beta_1 e^{\sqrt{-2}ct} + \beta_2 e^{-\sqrt{-2}ct}, \]

\[ \phi = -\frac{a\alpha_1}{\sqrt{2}c} e^{\sqrt{2}ct} + \frac{a\alpha_2}{\sqrt{2}c} e^{-\sqrt{2}ct} - \frac{b\beta_1}{\sqrt{-2}c} e^{\sqrt{-2}ct} + \frac{b\beta_2}{\sqrt{-2}c} e^{-\sqrt{-2}ct} \]
Case 1.8a, Dimension of group = 2

\[ M = C(x) + by \]

\[ \tau = \gamma = \xi_0 = 0, \quad \eta_0 = \beta_1 t + \beta_0, \quad \phi = -\frac{b \beta_1}{2} t^2 - b \beta_0 \]

with \( C_0 x^2 + ax + c_0 \neq C(x) \neq \frac{C_0}{x^2} + c_0 \).
Case 1.8b, Dimension of group = 2

\[ M = C(x) + cy^2 + by \]

\[ \tau = \gamma = \xi_0 = 0, \quad \eta_0 = \beta_1 e^{\sqrt{2}ct} + \beta_2 e^{-\sqrt{2}ct}, \]

\[ \phi = -\frac{b\beta_1}{\sqrt{2c}} e^{\sqrt{2}ct} + \frac{b\beta_2}{\sqrt{2c}} e^{-\sqrt{2}ct} \]

with \[ C_0x^2 + ax + c_0 \neq C(x) \neq \frac{C_0}{x^2} + cx^2 + c_0. \]
Finding Fundamental Solutions

The fundamental solutions are determined by

- Restricting the symmetry operator of the equation
  \[ u_t - \frac{1}{2} \Delta u + Mu = 0 \]
  to the initial value problem
  \[
  \begin{cases}
  u_t - \frac{1}{2} \Delta u + Mu = 0 \\
  u(x, 0) = \delta_\xi(x)
  \end{cases}
  \]

- Finding the invariant solutions to the restricted symmetry operators.
The following convention are used in the following slides

- **Sign convention**

\[ \pm = \begin{cases} + & \text{when } c > 0 \\ - & \text{when } c < 0 \end{cases} \]

- **Notation convention**

\[ \cot^\pm = \begin{cases} \coth(\sqrt{2ct}) & \text{when } c > 0 \\ \cot(\sqrt{-2ct}) & \text{when } c < 0 \end{cases} , \]

and similarly for \( \sin^\pm, \sec^\pm, \csc^\pm \) and \( \tan^\pm \).
Fundamental Solution - 1.1a

\[ u_t - \frac{1}{2} \Delta u + \left( \frac{c_0}{x^2} + by + c_0 \right) u = 0 \]

- **Fundamental Solution (Transition probability)**

\[ G^{(1a)} = K \left( \frac{x}{t} \right)^\kappa e^{-2 \frac{\xi x}{t} - c_0 t} \Phi \left( \kappa, 2\kappa, 2 \frac{\xi x}{t} \right) \times \]

\[ \exp \left\{ -\frac{(x - \xi)^2 + (y - \eta)^2}{2t} - \frac{b}{2}(y + \eta)t + \frac{b^2}{24}t^3 \right\} , \]

\[ \kappa = \frac{1 + \sqrt{1 + 8c_0}}{2} \]

- \( \Phi(\kappa, 2\kappa, z) \) is a confluent hypergeometric function of order \((\kappa, 2\kappa)\), i.e., \( \Phi \) satisfies the differential equation \( zv'' + (2\kappa - z)v' - \kappa v = 0 \), with the asymptotic behavior

\[ \Phi \to \frac{\Gamma(2\kappa)}{\Gamma(\kappa)} e^z z^{-\kappa} \text{ as } z \to +\infty. \]
\[ u_t - \frac{1}{2} \Delta u + \left( \frac{C_0}{x^2} + c(x^2 + y^2) + by + c_0 \right) u = 0 \]

Fundamental Solution (Transition probability)

\[
G^{(1b)} = K(x \csc^{\pm}) \kappa e^{\gamma x \csc^{\pm} - c_0 t \csc^{\pm} \Phi(\kappa, 2\kappa, -2\gamma x \csc^{\pm})} \times \\
\exp \left\{ -\sqrt{\frac{|c|}{2}} \cot^{\pm} (x^2 + \xi^2) \right\} \times \\
\exp \left[ -\cot^{\pm} \left\{ \sqrt{\frac{|c|}{2}} (y - \eta \sec^{\pm})^2 \pm \frac{b}{\sqrt{2|c|}} (1 - \sec^{\pm})(y - \eta \sec^{\pm}) \right\} \right] \times \\
\exp \left[ \frac{b^2}{4|c|} \left( \sqrt{\frac{2}{|c|}} (\csc^{\pm} - \cot^{\pm}) \pm t \right) - \left( \frac{b\eta}{\sqrt{2|c|}} \pm \sqrt{\frac{|c|}{2}} (\eta^2) \right) \tan^{\pm} \right]
\]
Fundamental Solution - 1.2a

\[ u_t - \frac{1}{2} \Delta u + (C(\theta) \frac{1}{r^2} + c_0)u = 0 \]

**Fundamental Solution (Transition probability) at \((0, 0)\)**

\[ G^{(2a)} = Ke^{-\cot \frac{x^2 + y^2}{2t}} l(\theta) \]

where \(l\) is a solution to

\[ l''''(\theta) = 2C(\theta)l(\theta) \quad \text{for } \theta \in [0, 2\pi] \]

with the boundary condition \( l(0) = l(2\pi) \)
\begin{itemize}
  \item $u_t - \frac{1}{2} \Delta u + (C(\theta) \frac{1}{r^2} + c(x^2 + y^2) + c_0)u = 0$
  \item Fundamental Solution (Transition probability) at $(0, 0)$
    \[ G^{(2b)} = Ke^{-\cot} \exp \left[ -\sqrt{\frac{|c|}{2}} \cot \pm (x^2 + y^2) \right] l(\theta), \]
    where $l$ is a solution to
    \[
    l''''(\theta) = 2C(\theta)l(\theta) \quad \text{for} \quad \theta \in [0, 2\pi]
    \]
  \end{itemize}

with the boundary condition $l(0) = l(2\pi)$
Fundamental Solution - 1.4a

- $u_t - \frac{1}{2} \Delta u + \left( \frac{C_0}{r^2} + c_0 \right) u = 0$

- Fundamental Solution (Transition probability) at $(0, 0)$

$$G^{(4a)} = \frac{K e^{-c_0 t}}{t} \left( \frac{\sqrt{x^2 + y^2}}{t} \right)^\lambda \Phi \left( 1 + \lambda, 1 + \lambda, -\frac{x^2 + y^2}{2t} \right)$$

- $\lambda = \sqrt{2C_0}$

- $\Phi$ is the incomplete Gamma function

$$\Gamma(-\lambda, z) = \int_z^\infty t^{-\lambda-1} e^{-t} dt.$$
\[ u_t - \frac{1}{2} \Delta u + \left( \frac{C_0}{r^2} + c(x^2 + y^2) + c_0 \right) u = 0 \]

Fundamental Solution (Transition probability) at \((0, 0)\)

\[
G^{(4b)} = \frac{Ke^{-c_0 t}}{\sin^\pm} \left( \frac{\sqrt{x^2 + y^2}}{\sin^\pm} \right)^\lambda e^{-\sqrt{\frac{|c|}{2}}(x^2 + y^2) \cot^\pm},
\]

where \( \lambda = \sqrt{2C_0} \)
\[ u_t - \frac{1}{2} \Delta u + (ax + by + c_0)u = 0 \]

Fundamental Solution (Transition probability)

\[ G^{(5a)} = Ke^{-c_0t} \frac{t}{t} \exp \left[ -\frac{(x-\xi)^2 + (y-\eta)^2}{2t} - \left( \frac{a}{2}(x+\xi) + \frac{b}{2}(y+\eta) \right) t + \frac{a^2 + b^2}{24} t^3 \right] \]
Fundamental Solution - 1.5b

- \( u_t - \frac{1}{2} \Delta u + (c(x^2 + y^2) + ax + by + c_0)u = 0 \)

- **Fundamental Solution (Transition probability)**

\[
G^{(5b)} = \frac{Ke^{-c_0 t}}{\sin^{\pm}} \times \\
\exp \left[ \frac{a^2 + b^2}{4|c|} \left( \sqrt{\frac{2}{|c|}} (\csc^{\pm} - \cot^{\pm}) \pm t \right) - \left( \frac{a\xi + b\eta}{\sqrt{2|c|}} \pm \sqrt{\frac{|c|}{2}} (\xi^2 + \eta^2) \right) \tan^{\pm} \right] \times \\
\exp \left[ -\cot^{\pm} \left\{ \sqrt{\frac{|c|}{2}} (x - \xi \sec^{\pm})^2 \pm \frac{a}{\sqrt{2|c|}} (1 - \sec^{\pm})(x - \xi \sec^{\pm}) \right\} \right] \times \\
\exp \left[ -\cot^{\pm} \left\{ \sqrt{\frac{|c|}{2}} (y - \eta \sec^{\pm})^2 \pm \frac{b}{\sqrt{2|c|}} (1 - \sec^{\pm})(y - \eta \sec^{\pm}) \right\} \right]
\]
Fundamental Solution - 1.7a

- \( u_t - \frac{1}{2} \Delta u + (cx^2 + ax + by + c_0)u = 0 \)

- Fundamental Solution (Transition probability)

\[
G^{(7a)} = \frac{Ke^{-co t}}{\sqrt{t} \sqrt{\sin \pm}} \exp \left[ \frac{a^2}{4|c|} \left( \sqrt{\frac{2}{|c|}} (\csc \pm - \cot \pm) \pm t \right) - \left( \frac{a \xi}{\sqrt{2|c|}} \pm \sqrt{\frac{|c|}{2}} \xi^2 \right) \tan \pm \right.
\]
\[
+ \frac{b^2 t^3}{24} - \frac{b}{2} (y + \eta) t \right] \times
\]
\[
\exp \left[ - \cot \pm \left\{ \sqrt{\frac{|c|}{2}} (x - \xi \sec \pm)^2 \pm \frac{a}{\sqrt{2|c|}} (1 - \sec \pm) (x - \xi \sec \pm) \right\}
\]
\[
\left. - \frac{(y - \eta)^2}{2t} \right].
\]
Fundamental Solution - 1.7b

- \( u_t - \frac{1}{2} \Delta u + (c(x^2 - y^2) + ax + by + c_0)u = 0 \)

- **Fundamental Solution (Transition probability)**

\[
G^{(7b)} = \frac{Ke^{-c_0 t}}{\sqrt{\sin^\pm \sin^\mp}} \times \\
\exp \left[ \frac{a^2}{4|c|} \left( \sqrt{\frac{2}{|c|}} (\csc^\pm - \cot^\pm) \pm t \right) - \left( \frac{a\xi}{\sqrt{2|c|}} \pm \sqrt{\frac{|c|}{2}} \xi^2 \right) \tan^\pm \right] \times \\
\exp \left[ \frac{b^2}{4|c|} \left( \sqrt{\frac{2}{|c|}} (\csc^\mp - \cot^\mp) \mp t \right) - \left( \frac{b\eta}{\sqrt{2|c|}} \mp \sqrt{\frac{|c|}{2}} \eta^2 \right) \tan^\mp \right] \times \\
\exp \left[ -\cot^\pm \left\{ \sqrt{\frac{|c|}{2}} (x - \xi \sec^\pm)^2 \pm \frac{a}{\sqrt{2|c|}} (1 - \sec^\pm) (x - \xi \sec^\pm) \right\} \right] \times \\
\exp \left[ -\cot^\mp \left\{ \sqrt{\frac{|c|}{2}} (y - \eta \sec^\mp)^2 \mp \frac{b}{\sqrt{2|c|}} (1 - \sec^\mp) (y - \eta \sec^\mp) \right\} \right]
\]
Chapter I: Closed form solution for quadratic and inverse quadratic models of the term structure

\[ u_t - \Delta u + Pu = 0, \]  

fundamental PDE or canonical form

where \( P \) is a potential. Typical examples of potentials arising in finance and in mathematical physics are:

- \( P = 0 \)  
  heat equation

- \( P = x_1^2 + \cdots + x_n^2 \)  
  harmonic oscillator

- \( P = \frac{\lambda}{|x|^2} \)  
  inverse square potential
The reason that the PDE \( u_t - \Delta u + Pu = 0 \) is so important derives from it’s being a prototype of a canonical form for a second order parabolic differential equation with variable coefficients. Indeed in the case of one spatial dimension it is the canonical form. I.e.

**Proposition 1** Any parabolic differential equation in one spatial variable

\[
  u_t - \frac{1}{2} a(x, t) u_{xx} - b(x, t) u_x + c(x, t) u = 0
\]

can, by a change of independent and dependent variables, be reduced to canonical form.
This result goes back to Lie and its importance was rediscovered in modern times by Bluman.
A Lie symmetry transformation is a transformation of the independent variables \(x, y, t\) and dependent variable \(y\) of the form

\[(x, y, t, u) \mapsto (\bar{x}, \bar{y}, \bar{t}, \bar{u})\]

\[\bar{x} = \alpha_1(x, y, t) \quad \bar{y} = \alpha_2(x, y, t) \quad \tau = \beta(t) \quad v = \gamma(\bar{x}, \bar{y}, \bar{t})\beta(\bar{x}, \bar{y}, \bar{t})\]

such that the equation

\[u_t - \frac{1}{2} \Delta u + Pu = 0\]

is has the same form in the new variables.
Lie’s symmetry analysis can be used to find similarity variables which allow one, depending on the size of the Lie symmetry group to either

- Reduce the PDE to an ODE.
- Or
- Integrate the PDE in closed form
An interesting application of Lie’s method is in expressing a particular solution of the PDE in closed form, namely the fundamental solution.

The application in mathematical finance is in expressing the fundamental solution of the zero coupon bond $P(t, T)$

$$P(t, T) = E \left[ \exp \left( \int_t^T -r(X_1, X_2, t) ds \right) \right]$$

in a situation where the pricing equation of $P(t, T)$ under the forward measure is for instance of the form

$$P_t - \frac{1}{2} \Delta P + a(t) X_1^2 + b(t) X_1 X_2 + c(t) X_2^2 = 0,$$

a special case of the so-called quadratic term structure.
It is possible to define a multi-asset local variance contract.

In the context of multi-dimensional local volatility models, the construction of the corresponding contingent claims involves solving for the fundamental solution of a variable coefficient elliptic operator.

The construction is only local. Can it be extended?

How many terms in the Hadamard expansion are needed to accurately approximate the fundamental solution in practice?

It is possible to extend Breeden and Litzenberger’s formula to $n$ assets given the knowledge of baskets with a continuum of strikes and of weights.

Since there are not sufficiently many weights and strikes, how can to best exploit the generalized Breeden and Litzenberger formula.