A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact

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Workshop on PDE and Mathematical Finance
KTH, Stockholm, August 15, 2005

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Introduction

• Classical financial market models:
  ▶ traders as price takers (small investors)
  ▶ continuous trading of unlimited amount (unbounded variation)

• Violation of these assumptions:
  ▶ large trades move the price of underlying asset (large investor)
  ▶ trading strategies constraints: transaction costs (finite variation strategies)

→ Liquidity risk or illiquid markets
• Various modelisation of liquidity risk:

  ▶ market microstructure literature: asymmetric information and insider trading:

    Kyle (85), Back (92)

  ▶ transaction costs literature and finite variation strategies:

    Davis-Norman (90), Jouini-Kallal (91), Longstaff (01)

  ▶ market manipulation literature and price impact:

    Cuoco-Cvitanic (98), Frey (98), Papanicolaou-Sircar (98), Bank-Baum (04), Cetin-Jarrow-Protter (04).
Our model adopts both perspectives of

- transaction costs and market manipulation literature
- inspired by recent works: Subramanian-Jarrow (01), He-Mamayski (05)

- Discrete trading (liquidity constraints faced in practice)
- Trade impact price: Prices are pushed up (resp. moved down) when investors buy (resp. sell) stock shares.
- Problem of optimal portfolio choice over a finite horizon:

Maximize the expected utility from terminal liquidation wealth under an economic solvency constraint.
1. Model and Problem formulation

- Uncertainty and information: \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), W\) one-dim BM, \([0, T]\).

- Money market account: constant interest rate \(r \geq 0\)

- Risky asset price: \(P = (P_t)_t\)

- Amount of money (cash holdings): \(X = (X_t)_t\)

- Cumulated number of asset shares: \(Y = (Y_t)_t\)

\(\rightarrow\) relevant state variables: \(Z = (X, Y, P)\)
Liability constraints: discrete trading is modelled by an *impulse control strategy* \( \alpha = (\tau_n, \zeta_n)_{n \geq 1} : \)

\[ \tau_1 \leq \ldots \tau_n \leq \ldots < T : \text{stopping times representing the intervention times of the investor. No trade at the terminal liquidation date} \ T. \]

\( \zeta_n \) valued in \( \mathbb{R} \) and \( \mathcal{F}_{\tau_n} \)-measurable: number of stocks purchased (if \( \zeta_n \geq 0 \)) or sold (if \( \zeta_n < 0 \)) at time \( \tau_n \)

\[ \rightarrow \text{Dynamics of } Y : \]

\[ Y_s = Y_{\tau_n}, \quad \tau_n \leq s < \tau_{n+1} \]

\[ Y_{\tau_{n+1}} = Y_{\tau_n} + \zeta_{n+1}. \]
• **Price impact**: positive function $Q(y, p)$: post-order price when the large investor trades a position of $y$ stock shares at a pre-order price $p$:

$Q(y, p)$ is nondecreasing in $y$

$Q(y, p) \geq (\text{resp.} \leq, =) p$ if $y \geq (\text{resp.} \leq, =) 0$

Throughout the paper:

$Q(y, p) = pe^{\lambda y}$, where $\lambda > 0$.

► **Transaction cost function** $\theta(y, p)$: (algebraic) cost for an investor with a position of $y$ stock shares when pre-trade price is $p$:

$\theta(y, p) = yQ(y, p)$.

► **Liquidation function** $\ell(y, p)$: value that an investor would obtained by liquidating immediately his stock position $y$ by a single block trade:

$\ell(y, p) = -\theta(-y, p) = yQ(-y, p)$. 
→ Price dynamics:

▶ In absence of trading:

\[ dP_s = P_s(bds + \sigma dW_s), \quad \tau_n \leq s < \tau_{n+1} \]

▶ When a trading \((\tau_{n+1}, \zeta_{n+1})\) occurs, the price jumps at time \(\tau_{n+1}\) by:

\[ P_{\tau_{n+1}} = Q(\zeta_{n+1}, P_{\tau_{n+1}}^-) \]

\[ \Delta P_s := P_s - P_{s^-} = Q(\Delta Y_s, P_{s^-}) - P_{s^-}. \]
Cash holdings dynamics:

- In absence of trading:
  \[
  dX_s = rX_s ds, \quad \tau_n \leq s < \tau_{n+1}
  \]

- When a trading \((\tau_{n+1}, \zeta_{n+1})\) occurs, this results in a variation of cash holdings by:
  \[
  X_{\tau_{n+1}} = X_{\tau_n^-} - \theta(\zeta_{n+1}, P_{\tau_{n+1}^-}) - k
  \]

\[\iff\]

\[
\Delta X_s := X_s - X_{s^-} = -\theta(\Delta Y_s, P_{s^-}) - k.
\]

\(k > 0\) fixed cost.
• Liquidation value and solvency constraint:

▶ Liquidation net wealth: for a state value \( z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \),

\[
L(z) = \max\{x + \ell(y, p) - k, x\}1_{y \geq 0} + (x + \ell(y, p) - k)1_{y < 0},
\]

* When \( \lambda = k = 0 \), \( L(z) = x + py \).

* Here, \( L \) is discontinuous on \((x, 0, p)\), but uppersemicontinuous.

▶ Solvency region:

\[
S = \left\{ z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* : L(z) > 0 \right\},
\]
and its boundary and closure:

\[
\partial S = \left\{ z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* : L(z) = 0 \right\} \quad \text{and} \quad \bar{S} = S \cup \partial S.
\]

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• Admissible controls $\leftrightarrow$ state constraints :

Given an initial state $(t, z) \in [0, T] \times \bar{S}$, an impulse control strategy $\alpha = (\tau_n, \zeta_n)_n$ is admissible, denoted:

$$\alpha \in A(t, z),$$

if the associated state process $Z = (X, Y, P)$ starting from $z$ at $t$ satisfies the solvency constraint:

$$L(Z_s) \geq 0, \quad t \leq s \leq T,$$

i.e.

$$Z_s = (X_s, Y_s, P_s) \in \bar{S}, \quad t \leq s \leq T.$$

First question :

$$A(t, z) \neq \emptyset!$$
Optimal investment problem:

- Utility function $U : \mathbb{R}_+ \rightarrow \mathbb{R}$, increasing, concave, $U(0) = 0$,

$$U(w) \leq Kw^\gamma, \quad \gamma \in [0, 1).$$

Notation:

$$U_L(z) = U(L(z)), \quad z \in \bar{S}.$$

Value function:

$$v(t, z) = \sup_{\alpha \in \mathcal{A}(t, z)} \mathbb{E} [U_L(Z_T)], \quad (t, z) \in [0, T] \times \bar{S}.$$

→ Impulse stochastic control problem under state constraint.
• Our main goal:
  ▶ Rigorous PDE characterization of the problem by a viscosity solutions approach
  ▶ Numerical results

• Related literature on impulse control
  ▶ General theory and link with QVHJBI: Bensoussan-Lions (82), Perthame (85), Oksendal-Sulem (04).
  ▶ Applications in finance: Jeanblanc-Shiryaev (95), Korn (98), Cadenillas-Zapatero (99), Oksendal-Sulem (01)
  ▶ Viscosity approach: Ishii K. (93), Tang-Yong (93)
2. QVHJBI associated to dynamic programming principle

\[ \min \left[ -\frac{\partial v}{\partial t} - \mathcal{L}v, \; v - \mathcal{H}v \right] = 0, \quad \text{on} \; [0, T) \times S \]

The time-space liquidation solvency region \([0, T) \times S\) is divided into

* A no-trade region

\[ \mathcal{N}T = \{(t, z) \in [0, T) \times S : v(t, z) > \mathcal{H}v(t, z)\}. \]

* An impulse trade region

\[ \mathcal{T} = \{(t, z) \in [0, T) \times S : v(t, z) = \mathcal{H}v(t, z)\}. \]
\( \mathcal{L} \) is the second order local operator:

\[
\mathcal{L}\varphi = rx \frac{\partial \varphi}{\partial x} + bp \frac{\partial \varphi}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2}.
\]

associated to the no-trade diffusion system:

\[
dZ^0_s = \begin{pmatrix}
  dX^0_s \\
  dY^0_s \\
  dP^0_s
\end{pmatrix} = \begin{pmatrix}
  rX^0_s ds \\
  0 \\
  P^0_s (b ds + \sigma dW_s)
\end{pmatrix}.
\]

\( \mathcal{H} \) is the impulse nonlocal operator:

\[
\mathcal{H}\varphi(t, z) = \sup_{\zeta \in C(z)} \varphi(t, \Gamma(z, \zeta)),
\]

with the impulse transaction function associated to jumps of \( Z \):

\[
\Gamma(z, \zeta) = (x - \theta(\zeta, p) - k, y + \zeta, Q(\zeta, p)),
\]

and the admissible impulse transaction set:

\[
C(z) = \{ \zeta \in \mathbb{R} : L(\Gamma(z, \zeta)) \geq 0 \}.\]
3. First technical difficulties

- Check that $\mathcal{A}(t, z) \neq \emptyset$. Not trivial due to the shape of the solvency region.

- Careful analysis of transaction set $\mathcal{C}(z)$ for $z \in \partial S$. Actually, we have the following picture on the boundary of the solvency region:
Figure 1: The solvency region when $k = 1, \lambda = 1$

Figure 2: The solvency region when $p < k$

Figure 3: The solvency region when $p > k$
Finiteness and bound of the value function!

Due to nonlinearity of price impact and transaction function: $Q(y, p) = pe^{\lambda y}$, $\theta(y, p) = p ye^{\lambda y}$, the liquidation net wealth may grow after transaction:

$$L(\Gamma(z, \zeta)) > L(z), \text{ for some } z \in \bar{S}, \zeta \in C(z).$$

We do not have the usual Merton bound as in transaction costs model!

Introduce a suitable “linearization” of the liquidation value:

$$\bar{L}(z) = x + \frac{p}{\lambda} (1 - e^{-\lambda y}).$$

Key (elementary) properties:

$$L(z) \leq \bar{L}(z)$$

$$\bar{L}(\Gamma(z, \zeta)) \leq \bar{L}(z) - k.$$
Proposition. The value function $v$ is well-defined on $[0, T] \times \bar{S}$, and we have

$$0 \leq v(t, z) \leq v_0(t, z) := \mathbb{E} \left[ U \left( \tilde{L} \left( Z_{T}^{0, t, z} \right) \right) \right]$$

$$\leq Ke^{\rho(T-t)}\tilde{L}(z)$$

$$\leq K \left( 1 + \left( x + \frac{p}{\lambda} \right)^{\gamma} \right) \in G_{\gamma}([0, T] \times \bar{S})$$

where $\rho$ is a positive constant s.t.

$$\rho > \frac{\gamma b^2 + r^2 + \sigma^2 r (1 - \gamma)}{1 - \gamma} \frac{1}{\sigma^2}.$$ 

Remark. Sharp upper bound: when $\lambda \to 0$ (no price impact), we find the Merton bound

$$v(t, z) \leq \mathbb{E} [U (X_{T}^{0, t, x} + yP_{T}^{0, t, p})] \leq Ke^{\rho(T-t)}(x + py)^{\gamma}.$$
• Discontinuity of the value function on the solvency boundary.

▶ $v$ is discontinuous on $[0,T) \times D_k$ and on $[0,T) \times (\partial^x_1 S \cap \partial^+_\ell S)$.

▶ But $v$ is continuous on $[0,T) \times D_0$:

$$\lim_{(t',z') \to (t,z)} v(t',z') = v(t,z) = 0, \quad \forall (t,z) \in [0,T) \times D_0.$$ 

• Terminal condition (discontinuity in $T$).

$$v(T^-, z) := \lim_{t \not\to T} v(t,z) = \max [U_L(z), HU_L(z)].$$

Remark:

$$v(T^-, z) \neq v(T, z) = U_L(z).$$
4. Viscosity property

- No homogeneity of the value function, even for power utility function
- Solvency region not convex, value function not convex, continuity in $S$ not direct

- Discontinuous viscosity solutions

- To handle state constraints related to solvency constraint, we use in fact the concept of \textit{constrained viscosity solutions}, introduced by Soner (86).
Theorem. $v$ is a constrained viscosity solution to the QVHJBI:

$$\min \left[ -\frac{\partial v}{\partial t} - \mathcal{L}v, \ v - \mathcal{H}v \right] = 0, \quad \text{in} \ [0, T) \times S$$

(1)

i.e.

(i) $v$ is a viscosity supersolution of (1) on $[0, T) \times S$

(ii) $v$ is a viscosity subsolution of (1) on $[0, T) \times \bar{S}$. 
5. Uniqueness result

- Comparison principle for constrained viscosity solution to the QVHJBI (1):

We can compare

a *viscosity subsolution* to (1) on $[0, T) \times \bar{S}$

to a *viscosity supersolution* of (1) on $[0, T) \times S$

once we can compare them

* at the terminal date (as usual in parabolic problems)

* but also on $[0, T) \times D_0$!
Theorem. Suppose $u \in G_{\gamma}([0, T] \times \bar{S})$ is an usc viscosity subsolution to (1) in $[0, T) \times \bar{S}$ and $w \in G_{\gamma}([0, T] \times \bar{S})$ is a lsc viscosity supersolution to (1) in $[0, T) \times S$ such that:

\[
\begin{align*}
  u(t, z) &\leq \liminf_{(t', z') \to (t, z)} w(t', z'), \quad \forall \ (t, z) \in [0, T) \times D_0, \\
  u(T, z) := \limsup_{(t, z') \to (T, z)} u(t, z') &\leq w(T, z) := \liminf_{(t, z') \to (T, z)} w(t, z'), \quad \forall \ z \in \bar{S}.
\end{align*}
\]

Then,

\[
  u \leq w \quad \text{on} \quad [0, T] \times S.
\]

Remark. One cannot hope to derive a comparison principle in the whole closed region $\bar{S}$ since it would imply the continuity of the value function on $\bar{S}$, which is not true.
**Corollary.** The value function $v$ is continuous on $[0, T) \times S$ and is the unique (in $[0, T) \times S$) constrained viscosity solution to (1) lying in $G_\gamma([0, T] \times \bar{S})$ and satisfying the boundary condition:

$$\lim_{(t', z') \to (t, z)} v(t', z') = 0, \quad \forall (t, z) \in [0, T) \times D_0,$$

and the terminal condition

$$v(T^-, z) := \lim_{(t, z') \to (T, z)} v(t, z') = \max [U_L(z), \mathcal{H}U_L(z)], \quad \forall z \in \bar{S}.$$
Main steps of the proof

Step 1. To deal with the impulse obstacle problem in the QVHJBI, produce a suitable perturbation of viscosity supersolution \( \rightarrow \) strict viscosity supersolution:

Given \( \nu > 0 \), consider the perturbation smooth function on \([0, T] \times \bar{S}\):

\[
\phi_\nu(t, z) = e^{\rho'(T-t)} \left[ \tilde{L}(z) \gamma' + \nu \left( \frac{e^{\lambda y}}{p} + pe^{-\lambda y} \right) \right],
\]

where \( \rho' \) is large enough. Let \( u \in G_\gamma([0, T] \times \bar{S}) \) be a viscosity subsolution and \( w \in G_\gamma([0, T] \times \bar{S}) \) be a viscosity supersolution. We perturb \( w \) by setting:

\[
w_\varepsilon = w + \varepsilon \phi_\nu, \quad \varepsilon > 0.
\]
Then,

$$\lim_{|z| \to \infty} (u - w_\epsilon)(t, z) = -\infty,$$

for any $\epsilon > 0$, and for any compact set $K$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^*$, $w_\epsilon$ is a strict viscosity supersolution to (1) on $[0, T) \times K$: there exists some constant $\delta$ (depending on $K$) s.t.

$$\min \left[ -\frac{\partial w_\epsilon}{\partial t} - Lw_\epsilon, w_\epsilon - Hw_\epsilon \right] \geq \delta \epsilon, \quad \text{in} \quad [0, T) \times K,$$

holds in the viscosity sense.
Step 2. General viscosity solution technique with specificities coming from

★ nonlocal impulse operator

★ boundary conditions: difficulties arising from the nonregularity of the solvency boundary (corners)

▶ For the boundary conditions, the idea is to build a smooth test function so that the minimum associated with the supersolution cannot be on the boundary.

Remark. The method of Soner (86) does not work here since it requires the continuity of the supersolution on the boundary, which is precisely not the case here!
Use a method of Barles (93). We (re)define \( w \) on \([0, T') \times \partial S\) by:

\[
w(t, z) = \liminf_{(t', z') \to (t, z)} w(t', z'), \quad \forall (t, z) \in [0, T') \times \partial S,
\]

so that \( w \) and \( w_m \) are still lsc on \([0, T] \times \bar{S}\). Consider \((t_0, z_0)\) s.t.

\[
(u - w_\varepsilon)(t_0, z_0) = \max_{[0, T] \times \bar{S}} (u - w_\varepsilon) > 0 \text{ (by contradiction)}
\]

and suppose from the terminal and boundary conditions that

\[
(t_0, z_0) \in [0, T) \times \partial S \setminus D_0.
\]
By (3), there exists \((t_n, z_n)_n \in [0, T) \times S\) converging to \((t_0, z_0)\) and s.t. 
\[w_\varepsilon(t_n, z_n) \to w_\varepsilon(t_0, z_0)\]

Consider the test function
\[
\varphi_n(t, t', z, z') = \frac{|t - t'|^2}{|t_n - t_0|} + \frac{|z - z'|^2}{|z_n - z_0|} + \left(\frac{d(z')}{d(z_n)} - 1\right)^4,
\]
where \(d(.)\) is the distance to \(\partial S\), which is smooth \(C^2\) on a neighborhood of \(z_0 \notin D_0\).

Take a maximum \((\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n)\) of \(\varphi_n(t, t', z, z')\). As usual, \(\varphi_n(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n) \to 0\) so that \((\hat{t}_n, \hat{t}'_n) \to (t_0, t_0)\), \((\hat{z}_n \hat{z}'_n) \to (z_0, z_0)\) and :
\[d(\hat{z}'_n) \geq \frac{1}{2} d(z_n) > 0, \quad \text{and so} \quad \hat{z}'_n \in S.
\]

Write the subsolution property of \(u\) at \((\hat{t}_n, \hat{z}_n) \in [0, T) \times \bar{S}\), the strict supersolution property of \(w_\varepsilon\) at \((\hat{t}'_n, \hat{z}'_n) \in [0, T) \times S\), and apply Ishii's lemma to get the required contradiction.
6. Further development

- More general price impact function $Q(y, p)$.

- Numerical resolution of the QVHJBI characterizing $v$, e.g. by iterated optimal stopping problems (see Chancelier, Oksendal, Sulem 02).

- Analyse liquidity effects on optimal portfolio and compare with the case of no price impact.