Optimal risk sharing for

law invariant monetary utility functions

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Optimal Risk Sharing is a classical topic.

Actuarial literature on Re-Insurance:

- Arrow (1963)
- Borch (1962)
- Bühlmann (1979, 1984)
- Gerber (1979)

... 

- Barrieu, El Karoui (2002-2005)
- Dana, Scarsini (2005)

A new ingredient is the increasing use of *Risk Measures* in the finance industry (Basel II).
Definition

A functional $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a convex risk measure if it is

- **monotone**, i.e., $X_1 \leq X_2 \Rightarrow \rho(X_1) \geq \rho(X_2)$

- **convex**, i.e., $\rho(\lambda X_1) + (1 - \lambda)\rho(X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2)$ for $0 \leq \lambda \leq 1$

- **cash invariant**, i.e., $\rho(X + \text{const}) = \rho(X) - \text{const}$
Example 1

Value at Risk $\text{VaR}_\alpha$:

$$\rho(X) = -\sup \{x \mid P[X \leq x] \leq \alpha\},$$

where $\alpha \in ]0,1[$, e.g., $\alpha = 5\%$.

This “risk measure” fails to be convex!
Example 2

Average Value at Risk $\text{AV@R}_\alpha$ or "expected shortfall":

$$\rho(X) = E \left[ X \bigg| X \leq \rho_{\text{V@R}}(X) \right].$$

Example 3

Standard Deviation Principle:

\[ \rho(X) = E[X] + \beta E \left[ (X - E[X])^2 \right]^{\frac{1}{2}} \]

where \( \beta \geq 0 \).

This “risk measure” fails to be monotone.
Example 4

Semi-Deviation Principle:

$$\rho(X) = \mathbb{E}[X] + \beta \mathbb{E} \left[ (X - \mathbb{E}[X])^2 \right]^{\frac{1}{2}}$$
Example 5

Entropic Risk Measure:

\[ \rho(X) = \frac{1}{\gamma} \ln \mathbb{E} \left[ e^{-\gamma X} \right], \]

for \( \gamma > 0. \)
Basic Question

Two (or \( n \)) economic agents, \( i = 1, 2 \), are endowed with risky portfolios \( X_1, X_2 \in L^\infty(\Omega, \mathcal{F}, P) \) and use risk measures \( \rho_1, \rho_2 \). What kind of risk exchange do we expect to happen?

More Formally

We look for \( \xi_1, \xi_2 \in L^\infty(\Omega, \mathcal{F}, P) \) such that \( X_1 + X_2 = \xi_1 + \xi_2 \) and such that the agents \( i = 1, 2 \) are “happier” with \( \xi_i \) then with \( X_i \) (where “happiness” is measured by the risk measures \( \rho_i \)).
Definition

A function $U : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a monetary utility function if $U$ is

- monotone, i.e., $X_1 \leq X_2 \Rightarrow U(X_1) \leq U(X_2)$

- concave,

- cash invariant, i.e., $U(X + \text{const}) = U(X) + \text{const}$
Obvious

$U$ is a monetary utility function $\iff \rho := -U$ is a convex risk measure.

What does it mean to be “happier” after the risk exchange for agents $i = 1, 2$ endowed with risky position $X_i \in L^\infty$ and a monetary utility function)?

We call a pair $\xi_1, \xi_2 \in L^\infty \times L^\infty$ an admissible allocation if $\xi_1 + \xi_2 = X_1 + X_2$. 
**Definition**

An admissible allocation \((\xi_1, \xi_2)\) is *Pareto optimal* if, for each admissible allocation \((\eta_1, \eta_2)\) with

\[
U_1(\eta_1) \geq U_1(\xi_1), \quad U_2(\eta_2) \geq U_2(\xi_2)
\]

implies that

\[
U_1(\eta_1) = U_1(\xi_1), \quad U_2(\eta_2) = U_2(\xi_2).
\]
Observation

For monetary utility functions $U_1, U_2$ and a Pareto optimal allocation $(\xi_1, \xi_2)$ we have that $(\xi_1 + \text{const}, \xi_2 - \text{const})$ is Pareto optimal too, for each $\text{const} \in \mathbb{R}$.

Definition

An admissible allocation $(\xi_1, \xi_2)$ is individually rational if

$$U_1(\xi_1) \geq U_1(X_1), \quad U_2(\xi_2) \geq U_2(X_2).$$
Proposition

Given $X_1, X_2$ and $U_1, U_2$ as above, there is $r \geq 0$, called the *rent of risk exchange*, such that, for every Pareto optimal admissible allocation $(\xi_1, \xi_2)$ we have

$$U_1(\xi_1) + U_2(\xi_2) = U_1(X_1) + U_2(X_2) + r$$

and such that there is an interval $[c_1, c_2] \subseteq \mathbb{R}$ with $c_2 - c_1 = r$ and such that $(\xi_1 + c, \xi_2 - c)$ is individually rational iff $c \in [c_1, c_2]$. 
Remark (speaking economically)

The search of an optimal (i.e. Pareto optimal and individually rational) risk exchange has two aspects: firstly the two agents have a common interest to find a Pareto optimal allocation \((\xi_1, \xi_2)\); secondly they have an adverse interest in fixing the price \(c\).
Example (one of the main results of the paper)

Suppose that $U_1(X) = -\rho_{\text{AV@R}}(X)$ and $U_2(.)$ is in a rather general class of monetary utility functions (including the “semi-deviation” as well as the “entropic” utility).

Assume also (for convenience only) that the law of the total risk $X = X_1 + X_2$ is diffuse.

Then there is a unique (up to a constant) Pareto optimal admissible allocation, namely

$$\xi_1 = (X - k)_-, \quad \xi_2 = X - (X - k)_-$$

for some $k \in \mathbb{R}$ (which can, in principle, be computed).
**General Approach**

Legendre-Fenchel transform:

\[ U_i^*(f) = \sup_{X \in L^\infty(\Omega, \mathcal{F}, P)} [U_i(X) - \langle X, f \rangle], \]

\( f \in L^1(\Omega, \mathcal{F}, P) \) (or, maybe, \( f \in L^\infty(\Omega, \mathcal{F}, P)^* \)).

Remark: The cash invariance and monotonicity of \( U_i \) implies that it is Lipschitz w.r. to \( \| . \|_\infty \). Hence one is sure that the duality theory works for the dual pair \( \langle L^\infty, (L^\infty)^* \rangle \).

Sup-Convolution:

\[ U_1 \Box U_2(X) = \sup_{\xi_1 + \xi_2 = X} U_1(\xi_1) + U_2(\xi_2). \]
Wellknown fact:

(i) \((U_1 \boxplus U_2)^* = U_1^* + U_2^*\)

(ii) \(f \in \text{supergrad}(U_i(X)) \iff -X \in \text{subgrad}(U_i^*(f))\).

Admitting the formula

\[
\text{subgrad}(U_1^* + U_2^*) = \text{subgrad}(U_1^*) + \text{subgrad}(U_2^*)
\]  \((*)\)

we have the following recipe to find a Pareto optimal allocation for a given total risk \(X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})\).
Find $f \in \text{supergrad}(U_1 \Box U_2)(X)$ so that

$$-X \in \text{subgrad} \left( (U_1 \Box U_2)^*(f) \right) = \text{subgrad} \left( U_1^*(f) + U_2^*(f) \right)$$

Supposing (*): $$= \text{subgrad} \left( U_1^*(f) \right) + \text{subgrad} \left( U_2^*(f) \right).$$

Now choose $-\xi_i \in \text{subgrad} \left( U_i^*(X) \right)$ such that $\xi_1 + \xi_2 = X$.

As $f \in \text{supergrad} \left( U_1^*(\xi_1) \right) \cap \text{supergrad} \left( U_2^*(\xi_2) \right)$ one quickly verifies that $(\xi_1, \xi_2)$ is Pareto optimal.

**Question**

Is (*) true in the present context?
Answer

In general, NO!

However, if we suppose that $U_i$ are law invariant then the answer is YES.

Definition

A function $U : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is law invariant if $\text{law}(X) = \text{law}(Y)$ implies $U(X) = U(Y)$. 
Theorem 1

A law invariant monetary utility function \( U : L^\infty \to \mathbb{R} \) is weak star lower semi-continuous. Hence the Legendre-Fenchel duality works for the dual pair \( \langle L^\infty, L^1 \rangle \).

Theorem 2

Let \( U_1, U_2 \) be law invariant monetary utility functions such that

\[
U_1 \Box U_2(0) = \sup \{ U_1(\xi) + U_2(-\xi) \mid \xi \in L^\infty \} < \infty.
\]

Then, for \( X \in L^\infty \), there exists a Pareto optimal allocation \((\xi_1, \xi_2) \in L^\infty \times L^\infty\).
Thank you
for your attention!