Power-law Price-impact Models and Stock Pinning near Option Expiration Dates

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Summary

Empirical evidence of stock pinning (Ni-Pearson-Poteshman, 2003)

Linear price-impact model (Avellaneda & Lipkin 2002)

Non-linear price-impact model (Ave., Kasyan & Lipkin, 2007)

Numerical simulation of pinning for different price-impact functions

Phase transition at p=1/2

Rigorous mathematical proofs for the different regimes.

Bibliography
Pinning on Option Expiration Dates

Option expiration Friday, (3rd Friday of the month).

Stock B pinned
Stock A did not
KO: Sep 18 to Oct 17 2003

10/17 Close=$45.05

Expiration
Statistical Evidence of Pinning


Authors: Sophie Xiaoyan Ni, Neil Pearson and Allen M. Poteshman

(U. of Illinois Urbana-Champaign)

Data 1. *Ivy DB (OptionMetrics)*
Jan 1996, Sep 2002: All stocks traded in US exchanges
All options traded in US exchanges
End of day bid-ask quotes, volume, open interest

2. *CBOE Statistics*
Open interest and trading volume, Jan 1996 to Dec 2001
4 Investor Categories: Market Makers, Firm Prop Traders,
Large Firm Clients, Discount Firm Clients
The U. Illinois Urbana study

- At least 80 expiration dates
- 4,395 optionable stocks on at least one date
- 184,449 optionable stock-expiration pairs
- 12,001 non-optionable stocks on at least one date
- 417,007 non-optionable stock-expiration pairs
Percentage of non-optionable stocks closing within $0.25 of an integer multiple of $5

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of **optionable** stocks closing within $0.25 of an integer multiple of $5

( Courtesy: Ni, Pearson & Poteshman)
Percentage of optionable stocks closing within $0.25 of a strike price

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of non-optionable stocks closing within $0.125 of an integer multiple of $5

(Courtesy: Ni, Pearson & Poteshman)
Percentage of **optionable** stocks closing within $0.125 of an integer multiple of $5

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of optionable stocks closing within $0.125 of a strike price

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Our Model: Feedback Due to Demand for Deltas

**Assumption 1.** Open Interest is unusually large

**Assumption 2.** Market-makers – professional delta-hedgers – are net very long options

**Proposed mechanism for pinning:**

Hedgers are net long options, hence long Gamma. They sell stock when it rises and buy stock when it falls.

Since the aggregate amount of stock required is large compared to typical daily trading volume, this drives the stock to the strike price
JDEC 2001 Mar 10
Put & Call Open Interest

Average traded vol in stocks = 1MM shares

Notional number of shares corresponding to OI = 5.6 MM shares
In search for an explanation…

JDEC in March 2001

Large sale of options on this day
Taking into account demand for stock: Price-Impact Functions

\[
\frac{dS}{S} \propto E\left(\frac{D}{<V>}\right)^p \quad \frac{D}{<V>} \gg 1
\]

- $p=0.22$ Farmer, Lillo, Mantegna
- $p=0.5$ X. Gabaix
- $p=1$ linear model, (A. & Lipkin)
- $p=1.5$ convex model (Bouchaud, …)

Choice of $p$ is a fundamental question in Econophysics.
Linear Model (A & Lipkin, 2002)

Price-Demand Elasticity Eq.

\[
\frac{\Delta S}{S} \propto E \cdot \frac{D}{\|D\|}
\]

Price-response due to demand for deltas

\[
\frac{\Delta S}{S} \propto \frac{E.OI}{\|D\|} \Delta \delta
\]

\[
\delta = B. - S. \Delta \text{ for one option}
\]
Estimating the Demand for Deltas using Black-Scholes

$$\Delta \delta = \frac{\partial \delta}{\partial t} \, dt, \quad \tau = T - t$$

$$\delta = 2N(d_1), \quad d_1 = \frac{1}{\sigma \sqrt{\tau}} \left( \ln \left( \frac{S}{K} \right) + \left( \mu + \frac{\sigma^2}{2} \right) \frac{\sqrt{\tau}}{2} \right)$$

From Black-Scholes

$$y = \ln \left( \frac{S}{K} \right), \quad a = \mu + \frac{\sigma^2}{2},$$

$$\frac{\partial \delta}{\partial t} = -\frac{1}{\sqrt{2\pi}} \frac{y - a \tau}{\sigma \tau^{3/2}} e^{\frac{(y+a\tau)^2}{2\sigma^2\tau}}$$
Dynamics for Stock Price

\[ \frac{dS}{S} = \frac{E.OI}{\| D \|} \frac{\partial \delta}{\partial t} dt + \sigma dW \]

\[ y = \ln \left( \frac{S}{K} \right) \]

\[ dy = - \frac{E.OI}{\| D \| \sqrt{2\pi}} \cdot \frac{y - a(T - t)}{\sigma(T - t)^{3/2}} e^{-\frac{(y + a(T-t))^2}{2\sigma^2(T-t)}} dt + \sigma dW \]

- `coupling constant`
- restoring force
- bounded support
- noise
Dimensionless Variables

\[ z = \frac{y}{\sigma \sqrt{T}}, \quad s = \frac{t}{T}, \quad z_0 = \frac{y_0}{\sigma \sqrt{T}} = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{S_0}{K} \right) \]

\[ \alpha = \frac{a \sqrt{T}}{\sigma}, \quad \beta = \frac{E.OI}{\langle |D| \rangle \sqrt{2\pi \sigma^2 T}} \]

\[ d\zeta = -\frac{\beta(z - \alpha(1-s))}{(1-s)^{3/2}} e^{-\frac{(z+\alpha(1-s))^2}{2(1-s)}} ds + d\bar{W} \]
Dimensionless Model (alpha=0) for Linear Price-Impact Function

\[ dz = -\frac{\beta \cdot z}{(1-s)^{3/2}} e^{-\frac{z^2}{2(1-s)}} ds + dW \]

Linear restoring force with increasing coupling with time and compact support.
The Potential Well

Price experiences a force that becomes stronger, more localized near expiration.

\[ z = \ln(S/K)/(\sigma \sqrt{\tau}) \]

\[ \frac{dz}{ds} = -\frac{z}{(1 - s)^{3/2}} e^{-\frac{z^2}{2(1-s)}} \quad (\alpha = 0, \beta = 1) \]
Solution of the Avellaneda-Lipkin linear response model \((p=1)\)

Assume \(\text{Alpha}=0\)

Forward Fokker-Planck equation:

\[
\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{\beta z}{\tau^{3/2}} e^{-\frac{z^2}{2\tau}} \frac{\partial F}{\partial z} = 0, \quad \tau = 1 - s
\]

Look for solution of the form:

\[
F(z, s) = \exp\left(\frac{1}{\sqrt{\tau}} \phi\left(\frac{z}{\sqrt{\tau}}\right)\right), \quad \phi(\zeta) \text{ unknown}, \quad \zeta = \frac{z}{\sqrt{\tau}}
\]
ODE for the `Phase Function' (WKB)

\[
\frac{\phi + \zeta \phi' + \phi''}{2\tau^{3/2}} + \frac{(\phi')^2 - 2\beta \zeta \phi' e^{-\frac{s^2}{2}}}{2\tau^2} = 0
\]

\[O(\tau^{-2}) \quad \quad (\phi')^2 - 2\beta \zeta \phi' e^{-\frac{s^2}{2}} = 0 \quad \text{Eikonal Equation}
\]

\[\therefore \quad \phi(\zeta) = -2\beta e^{-\frac{s^2}{2}} + c\]

\[O(\tau^{-3/2}) \quad \phi + \zeta \phi' + \phi'' = c \quad c = 0\]

\[F(z, s) = \exp \left[ -\frac{2\beta}{\sqrt{1-s}} e^{-\frac{z^2}{2(1-s)}} \right] \quad \text{Exact solution of the FFP Equation!} \]
A Formula for the Pinning Probability

\[ P(z, s) = 1 - \exp\left[ -\frac{2\beta}{\sqrt{1-s}} e^{-\frac{z^2}{2(1-s)}} \right] \]

Satisfies:

\[
\begin{align*}
\lim_{s \to 1^+} P(z, s) &= 0 \\
\lim_{s \to 1^+} P(0, s) &= 1
\end{align*}
\]

\[
\text{Prob}(z(1) = 0 \mid z(0) = z_0) = 1 - e^{-2\beta e^{-\frac{z_0^2}{2}}}
\]
Pinning Probability: Dependence on Beta

\[ \beta = \frac{E.OI}{\|D\|} \frac{1}{\sqrt{2\pi\sigma^2T}} \]

- Increases with OI
- Decreases with volat, expiration
- Decreases with the distance to strike
Pinning probability: dependence on $z_0$

$z = \frac{\ln(S/K)}{(\text{vol} \times \sqrt{T})}$

$\text{Probability}$

$\text{Alpha} = 0$
$\text{Beta} = 0.1$
Dimensionless Model for Power-Law Price-Impact Function (p>0)

\[
\frac{dS}{S} \propto -\text{const.} \frac{\partial \delta}{\partial t}^p \text{sign}\left(\frac{\partial \delta}{\partial t}\right) dt + \sigma dW
\]

\[
dz = -\beta \cdot |z|^p \text{sign}(z) e^{-\frac{pz^2}{2(1-s)^{3p/2}}} ds + dW
\]

Dimensionless eq. without irrelevant drift terms (alpha=0).
Pinning under non-linear price-impact models

Proposition:

(i) If $p \leq 1/2$, there is no pinning, i.e. $P[z(1)=0 | z(0)=z] = 0$, for all $z$.

(ii) If $p > 1/2$ pinning occurs with finite probability ($<1$) and

\[
\ln P(z(1) = 0 | z(0) = z) \propto - \frac{C(\beta, z)}{2p-1}
\]

\[
P_{pin} \geq e^{-\frac{c}{2p-1}}, \quad p > 1/2
\]
Calculation of Pinning Probabilities by MC Simulation (Gennady Kasyan)

Smooth fit near $p=0.5$
**Absence of Pinning for p<1/2 (I)**

\[
dz = -\frac{\beta}{\tau^p} \Omega \left( \frac{z}{\sqrt{\tau}} \right) dt + dW
\]

\[
\Omega(r) = |r|^p \text{sg}(r) e^{\frac{pr^2}{2}}
\]

**Novikov condition:** If we have

\[
E^W \left\{ e^{\int_0^1 (\text{drift})^2 dt} \right\} < \infty
\]

the measure induced by the z-process in path-space is absolutely continuous with respect to Wiener Measure.
Absence of Pinning for $p < 1/2$ (II)

Verify that Novikov condition holds:

$$E^W_e \left\{ e \int_0^1 (\text{drift})^2 dt \right\} = E^W_e \left\{ e \int_0^1 \frac{\beta^2}{\tau^2 p} \Omega^2 (\zeta_t) dt \right\}$$

$$< E^W_e \left\{ e \left\| \Omega \right\|_{L_\infty}^2 \beta^2 \int_0^1 \frac{dt}{(1-t)^2 p} \right\}$$

$$< \infty, \quad \text{if} \quad p < 1/2$$
Self-similarity approach (RG)

Kasyan (2007) : exploit the self-similarity properties of the model

\[ z' = az \]
\[ \tau' = a^2 \tau \quad \therefore \quad t' = (1 - a^2) + a^2 t \]

\[ d\tau' = -\frac{\beta a^{2p-1}}{(\tau')^p} \Omega \left( \frac{z'}{\sqrt{\tau'}} \right) dt' + dW \]

\[ \beta' = \beta a^{2p-1} \]
Absence of pinning at p=1/2

1. Chapman-Kolmogorov

\[ P^\beta [ z(1) = 0 | z(0) = 0 ] = \int \pi(z',0.5)P^\beta [ z(1) = 0 | z(0.5) = z' ]dz' \]

2. Renormalization group

\[ P^\beta [ z(1) = 0 | z(0.5) = z' ] = P^\beta [ z(1) = 0 | z(0) = z' \sqrt{2} ] \]

\[ P^\beta [ z(1) = 0 | z(0) = 0 ] = \int \pi(z',0.5)P^\beta [ z(1) = 0 | z(0) = z' \sqrt{2} ]dz' \]

\[ < \left( \int \pi(z',0.5)dz' \right)P^\beta [ z(1) = 0 | z(0) = 0 ] = P^\beta [ z(1) = 0 | z(0) = 0 ] \]

This gives a contradiction. The pinning probability must be zero.

Note: we used monotonicity in z (maximum principle).
Vanishing of Pinning Probability at $p=0.5$ (log scale)
Log probability vs. $1/(p-1/2)$
Sketch of proof of $\ln P \sim -\frac{C}{(2p-1)}$
(G. Kasyan, 2007)
Estimating the probability of remaining inside the parabola

\[ \tau_n = 1 / 4^n, \quad n = 0, 1, 2... \]

\[ P^\beta[z(1) = 0] \geq P^\beta \left\{ |z(0)| < 1; |z(1 - \tau_1)| < 1 / 2; ... |z(1 - \tau_n)| < 1 / 2^n; ... \right\} \]

\[ = P^\beta \left\{ |z(1 - \tau_n)| < 1 / 2^n; n = 0, 1, 2... \right\} \]

\[ \geq \prod_{n=1}^{\infty} \inf_{z < 2^{-n+1}} P^\beta \left\{ |z(1 - \tau_n)| < 1 / 2^n \left| z(1 - \tau_{n-1}) \right| = z \right\} \]

\[ = \prod_{n=1}^{\infty} \inf_{z < 1} P^\beta \cdot 2^{2p-1} \left\{ \left| z(3/4) \right| < 1 / 2 \left| z(0) \right| = z \right\} \]

\[ = \prod_{n=1}^{\infty} p(\beta \cdot 2^{n(2p-1)}) \]

Probabilities of transitioning between time 0 and 3/4 with different values of the coupling coefficient.
Exit from parabolic region

\[
P^\beta \left\{ \left| z(3/4) \right| > 1/2 \left| z(0) \right| < 1 \right\} \approx e^{-C\beta} \quad , \quad \beta \gg 1
\]
Most likely exit path

Follow the flow (to $z=0$) for time $\tau - 1/\beta$

Go against (large) flow for time $1/\beta$
Large Deviations Estimate

\[ P(\gamma) \approx \exp \left\{ -\int_0^\tau \left| \frac{d\gamma}{ds} - v(\gamma, s) \right|^2 ds \right\} \]

\('Action' asymptotics
(Varadhan, Azencott, etc.)

\approx \exp \left\{ -\int_{\tau-1/\beta}^\tau \left| \frac{d\gamma}{ds} \right|^2 ds + \beta \frac{\gamma^p}{(1-s)^{3p/2}} e^{-\frac{\gamma^2}{2(1-s)}} \right\}

\approx \exp \left\{ -C \int_{\tau-1/\beta}^\tau \beta^2 ds \right\}

because \quad \frac{d\gamma}{ds} \approx \beta

\approx \exp\{-C\beta\} \quad \text{for} \quad \beta \gg 1
Lower bound for pinning probability

\[
P[z(1) = 0] \geq \prod_{n=1}^{\infty} P^{\beta 2^n (2^{p-1})} \left\{ z(3/4) < 1/2 \ | \ z(0) < 1 \right\}
\]

\[
\geq \prod_{n=1}^{\infty} \left( 1 - C_1 e^{-2 \beta 2^n (2^{p-1})} \right)
\]

\[
= e^{\sum_{n=1}^{\infty} \ln \left( 1 - C_1 e^{-2 \beta 2^n (2^{p-1})} \right)}
\]

\[
\approx \exp \left( -C_1 \sum_{n=1}^{\infty} e^{-2 \beta 2^n (2^{p-1})} \right)
\]

Re-scaled betas!

Large deviations estimate

\[
\ln P[z(1) = 0] \geq -C_1 \sum_{n=1}^{\infty} e^{-2 \beta 2^n (2^{p-1})} \approx -C_1 \sum_{n=1}^{\infty} e^{-2 \beta \left( 1 + n \ln 2 \cdot (2 \cdot 2^{p-1}) \right)}
\]

\[
\approx - \frac{C_3}{2p-1} \quad \text{\((2p-1 \ll 1)\)}
\]
Conclusions

- Pinning of stock prices at strikes near expiration dates can be explained in terms of price-impact due to demand for deltas by hedgers.

- Modeling the demand for deltas using power-law price-impact functions such as those used in Econophysics gives rise to SDEs that generalize the Avellaneda-Lipkin linear model.

- The price impact function
  \[ \frac{dP}{P} \propto Q^p \]
  produces pinning at the strike if and only if \( p > 1/2 \).

- Models of price-impact with \( p > 0.5 \) are therefore more likely to be observed in practice, since they are consistent with the observable phenomenon of pinning.
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