Nonlinear option pricing models for illiquid markets: invariant properties and solutions

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Pricing equations in non perfectly liquid markets

Three different frameworks for modeling illiquid markets: transaction-cost models; reduced-form SDE-models; reaction-function or equilibrium models.

- Cetin, Jarrow & Protter 2004
- Musiela & Zariphopoulou 2004 (indifference prices)

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'Quadratic' transaction costs models

Cetin, Jarrow & Protter 2004

The **fundamental** stock price process \( S_t^0 \)

\[
dS_t^0 = \sigma S_t^0 dW_t + \mu S_t^0 dt, \mu \in \mathbb{R}, \sigma > 0, \tag{1}\]

and the transaction price \( \hat{S}_t(\alpha) = e^{\rho \alpha} S_t^0, \rho > 0 \) to be paid at \( t \) for trading \( \alpha \) shares. → transaction costs are proportional to the quadratic variation of the stock trading strategy \( (\Phi_t, \eta_t)_{t>0} \) with the value \( V_t = \Phi_t S_t^0 + \eta_t \) (in Markovian case \( \Phi_t = \phi(t, S_t^0) \)).

Let \( u(t, S_t^0) \) gives the value of a self-financing trading strategy with the stock position \( \phi(t, S_t^0) \) then \( \phi = u_S \) and

\[
    u_t + \frac{1}{2} \sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0, \quad u(T, S) = h(S), S \geq 0.
\]
Reduced-form SDE models

Let $\phi$ represent (a semimartingale) the stock trading strategy of the large investor, $\phi_t = \phi(t, S_t)$, the stock price satisfies the SDE

$$dS_t = \sigma S_t dW_t + \rho S_t d\phi_t, \quad \rho > 0.$$  \hspace{1cm} (2)

If the option hedger uses a particular trading strategy $\phi$ by $S^\phi$, then applying the Itô formula

$$dS^\phi_t = v^\phi(t, S^\phi_t) S^\phi_t dW_t + b^\phi(t, S^\phi_t) S^\phi_t dt$$

with adjusted volatility $v^\phi(t, S) = \frac{\sigma}{1 - \rho S^\phi S(t, S)}$.

A portfolio with value $V_t$ is termed self-financing, if satisfies the equation $dV_t = \phi_t dS^\phi_t$. Suppose $V_t = u(t, S_t^\phi)$ then $\phi = u_S$ and

$$u_t + \frac{1}{2} \frac{\sigma^2}{(1 - \rho S u_{SS})^2} S^2 u_{SS} = 0.$$  \hspace{1cm} (3)

where $\phi_t = u_S$.

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Equilibrium or reaction-function models

Introduce: a smooth reaction function $\psi$ gives the equilibrium stock price $S_t$ at time $t$ as a function of some fundamental value $F_t$ (follows geometric Brownian motion with a volatility $\sigma$) and the stock position of the large investor.

The normalized stock demand of the ordinary investors at time $t$ is $D(F_t, x)$, where $x$ is the proposed price of the stock. The normalized stock demand of the large investor is $\rho \phi_t$; $\rho \geq 0$ is measure of the size of the traders’s position relative to the total supply of the stock.

The equilibrium price $S_t = \psi(F_t, \rho \phi_t)$ is determined by the market clearing condition $D(F_t, S_t) + \rho \phi_t = 1$.

Let $\psi((F_t, \rho \phi_t) = F_t g(\rho \phi_t)$, then

$$dS_t = g(\rho \phi(t, S_t)) dF_t + \rho F_t g_\alpha(\rho \phi(t, S_t)) \phi_S(t, S_t) dS_t + b(t, S_t) dt.$$

For the value function $u(t, S)$ of a self-financing strategy

$$u_t + \frac{1}{2} \frac{\sigma^2}{\left(1 - \rho \frac{g_\alpha}{g(\rho S S)} S u_{SS}\right)^2} S^2 u_{SS} = 0.$$
Nonlinear Black-Scholes equations

The listed nonlinear PDEs, are all of the form

\[ u_t + \frac{1}{2} \sigma^2 v(S, \rho u_S, \rho u_{SS}) S^2 u_{SS} = 0, \]

where \( v(S, 0, 0) = 1 \), \( \rho \) is usually considered as a small parameter.

In other models (like Musiela & Zariphopoulou 2004) prices have been typically represented as solutions of quasi-linear PDEs of the following form

\[ u_t + \sigma(S, t) u_{SS} + F(\rho u_S, \rho u, S, t) = 0. \]
A model of a hedge-cost 
for a derivative product in illiquid markets

\[ u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{ \left( 1 - \rho S \lambda(S) u_{SS} \right)^2} = 0, \]

- where \( S \in [0, \infty), \ t \in [0, T], \)
- \( S \) denotes the price of the underlying asset
- \( u(S, t) \) denotes the hedge-cost of the claim with later defined payoff.

The values of \( \rho \) and \( \lambda(S) \) might be estimated from the observed option prices. Typically \( \rho \in (0, 1) \) and the function \( \lambda(S) \) can be piece-wise approximated by power functions.

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Jet bundle

Let we have two sets of variables: \( x_1, x_2 \) are independent variables, \( u \) is a dependent variable. Let \( (x_1, x_2) \in X \), and \( u \in U \) and \( M = X \times U \) be a base space.

We consider the spaces:
- \( U^{(1)} \) - the space of all first derivatives of \( u \) with respect to \( x_1, x_2 \),
- \( U^{(2)} \) - the space of all second derivatives of \( u \) with respect to \( x_1, x_2 \)

**Definition** The 2–nd prolongation of \( U \) is \( U^{(2)} = U \times U^{(1)} \times U^{(2)} \).

**Definition** The 2–nd jet space of \( M \), or the jet space of order 2, is \( M^{(2)} = X \times U \times U^{(1)} \times U^{(2)} = X \times U^{(2)} \).

This also called 2–th prolongation of \( M \).

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Differential equations

Let an 2–nd order differential equation in 2 independent and one dependent variable $u$ be given

$$\Delta(x, u^{(k)}) = \Delta(x_1, x_2, u, u_{x_1}, u_{x_2}, \ldots, u_{x_2x_2}) = 0,$$

$\Delta$ is a smooth map from the jet space $M = X \times U^{(2)}$ to some $l$–dimensional Euclidean space

$$\Delta : X \times U^{(2)} \to \mathbb{R}^l.$$

which defines the subvariety

$$L_\Delta = \{(x, u^{(2)}) : \Delta(x, u^{(2)}) = 0\} \subset X \times U^{(2)}.$$

The symmetry group $G_\Delta$ of $\Delta$ will be defined by

$$G_\Delta = \{g \in \text{Diff}(M^{(2)}) | \ g : L_\Delta \to L_\Delta\}.$$
Prolongation

Let us now consider a Lie-point vector field on $M$

$$V = \xi^1(x, u) \frac{\partial}{\partial x_1} + \xi^2(x, u) \frac{\partial}{\partial x_2} + \phi(x, u) \frac{\partial}{\partial u},$$

where $\xi^i(x, u)$ and $\phi(x, u)$ are smooth functions of their arguments, $V \in \text{Diff}(M)$. Assume there exist infinitesimal generators of an action $g \in G_\Delta$.

A Lie group of transformations acting on the base space $M$ will induce a transformation on $M^{(2)}$. 
The corresponding algebra $\mathcal{D}iff_{\Delta}(M^{(k)})$ will be composed of vectors

$$pr^{(k)}V = \xi^1(x, u)\frac{\partial}{\partial x_1} + \xi^2(x, u)\frac{\partial}{\partial x_2} + \phi(x, u)\frac{\partial}{\partial u}$$

$$+ \phi^{x_1}(x, u)\frac{\partial}{\partial u_{x_1}} + \phi^{x_2}(x, u)\frac{\partial}{\partial u_{x_m}}$$

$$+ \phi^{x_1x_1}(x, u)\frac{\partial}{\partial u_{x_1x_1}} + \phi^{x_1x_2}(S, t, u)\frac{\partial}{\partial u_{x_1x_2}} + \phi^{x_2x_2}(x, u)\frac{\partial}{\partial u_{x_2x_2}},$$

The symmetry algebra $\mathcal{D}iff_{\Delta}(M^{(2)})$ of the second order differential equation $\Delta = 0$ can be found as a solution of the defining equation

$$pr^{(2)}V(\Delta) = 0 \ (mod(\Delta = 0)). \quad (4)$$
Assume we found the infinitesimal generator

\[ V = \xi^1(x, u) \frac{\partial}{\partial x_1} + \xi^2(x, u) \frac{\partial}{\partial x_2} + \phi(x, u) \frac{\partial}{\partial u} \]

of an one parametric group \( G_\Delta \) acting on \( L_\Delta \). A local invariant \( \eta \) of \( G_\Delta \) is a solution of

\[ V(\eta) = \xi^1(x, u) \frac{\partial \eta}{\partial x_1} + \xi^2(x, u) \frac{\partial \eta}{\partial x_2} + \phi(x, u) \frac{\partial \eta}{\partial u} = 0. \] (5)

The general solution of this equation can be found by integrating the corresponding characteristic system

\[ \frac{dx^1}{\xi^1(x, u)} = \frac{dx^2}{\xi^2(x, u)} = \frac{du}{\phi(x, u)}. \]
A symmetry group $G_\Delta$ of equation $\Delta(w) = 0$

The equation under investigation is

$$\Delta(S, t, u, u_S, u_t, u_{SS}, u_{St}, u_{tt}) = u_t + \frac{\sigma^2 S^2 u_{SS}}{2 (1 - \rho S \lambda(S) u_{SS})^2} = 0.$$ 

the corresponding symmetry group will be defined by

$$G_\Delta = \{ g \in \text{Diff}(M^{(2)}) | \ g : L_\Delta \to L_\Delta \},$$

where $L_\Delta$ is a solution manifold defined by

$$L_\Delta = \{ w \in M^{(2)} | \Delta(w) = 0 \} \subset M^{(2)}.$$ 

The symmetry algebra $\mathcal{Diff}_\Delta(M)$ of the PDE $\Delta(w) = 0$ can be found as a solution of the defining equations

$$pr^{(2)}V(\Delta) = 0 \ (mod(\Delta = 0)).$$
The Lie algebra for a special case

Let the function $\lambda(S)$ has a special form

$$\lambda(S) \equiv \omega S^k, \quad \omega, k \in \mathbb{R}.$$ 

Now the symmetry algebra $\mathcal{D}_{\Delta}(M)$ admits four generators

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S \frac{\partial}{\partial u},$$

$$V_3 = \frac{\partial}{\partial u}, \quad V_4 = S \frac{\partial}{\partial S} + (1 - k) u \frac{\partial}{\partial u},$$

with commutator relations

$$[V_1, V_2] = [V_1, V_3] = [V_1, V_4] = [V_2, V_3] = 0,$$

$$[V_2, V_4] = -k V_2, \quad [V_3, V_4] = (1 - k) V_3.$$
The symmetry group of $\Delta(w) = 0$

To find the finite transformations for the solutions of equation $\Delta(w) = 0$ corresponding to this symmetry group we just integrate the system of ordinary differential equations

$$
\frac{d\tilde{S}}{d\epsilon} = \xi(\tilde{S}, \tilde{t}, \tilde{u}), \quad \frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{S}, \tilde{t}, \tilde{u}), \quad \frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{S}, \tilde{t}, \tilde{u}),
$$

with initial conditions

$$
\tilde{S}|_{\epsilon=0} = S, \quad \tilde{t}|_{\epsilon=0} = t, \quad \tilde{u}|_{\epsilon=0} = u.
$$

The solution of this system in the general case is

$$
\tilde{S} = S, \quad \tilde{t} = t + a_2 \epsilon,
$$

$$
\tilde{u} = u + a_3 S \epsilon + a_4 \epsilon, \quad \epsilon \in (-\infty, \infty)
$$
The symmetry group of $\Delta(w) = 0$ in the special case

$$\lambda(S) = \omega S^k$$

If the function $\lambda(S)$ has a special form $\lambda(S) = \omega S^k$ we obtain a larger symmetry group

$$\tilde{S} = S e^{a_1 \epsilon}, \quad \tilde{t} = t + a_2 \epsilon, \quad \epsilon \in (-\infty, \infty)$$

if $k \neq 0$, $k \neq 1$ then the transformed value of the function $u(S, t)$ is given by

$$\tilde{u} = u e^{a_1 (1-k) \epsilon} + \frac{a_3}{a_1 k} S e^{a_1 \epsilon} (1-e^{-a_1 k \epsilon}) + \frac{a_4}{a_1 (1-k)} (e^{a_1 (1-k) \epsilon} - 1), \quad k \neq 0, k \neq 1$$

if $k = 0$

$$\tilde{u} = u e^{a_1 \epsilon} + a_3 S e^{a_1 \epsilon} + \frac{a_4}{a_1} (e^{a_1 \epsilon} - 1),$$

in case $k = 1$

$$\tilde{u} = u + \frac{a_3}{a_1} S (e^{a_1 \epsilon} - 1) + a_4 \epsilon.$$
The invariants of the symmetry group $G_{\Delta}$

In the general case the symmetry group $G_{\Delta}$ is very poor and we can obtain just following invariants

$$inv_1 = S, \quad inv_2 = u - (a_3 S + a_4)/a_2, \quad a_2 \neq 0.$$  

In the special case $\lambda(S) = \omega S^k$ the symmetry group admits two functionally independent invariants of the form

$$inv_1 = \log S + at, \quad a = a_1/a_2, \quad a_2 \neq 0, \quad inv_2 = u S^{(k-1)}.$$  

These invariants can be used as new independent and dependent variables to reduce the partial differential equation with the special function $\lambda(S')$ to an ordinary differential equation.
Reduction of $\triangle = 0$ in a special case to an ordinary differential equation

The equation under investigation is now

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1 - b S^{k+1} u_{SS})^2} = 0$$

with the constant $b = \rho \omega$, $\rho \in (0, 1)$, $\omega \neq 0$. Let us now introduce new invariant variables

$$z = \log S + at, \ a \neq 0, \ v = u S^{(k-1)}.$$

After this substitution the equation will be reduced to an ordinary differential equation

$$a v_z + \frac{\sigma^2}{2} \frac{v_{zz} + (1 - 2k)v_z - k(1 - k)v}{(1 - b(v_{zz} + (1 - 2k)v_z - k(1 - k)v))^2} = 0, \ a, b \neq 0.$$
The case $\lambda(S') = \omega S$

The equation takes the form

$$av_z + \frac{\sigma^2}{2} \frac{v_{zz} - v_z}{(1 - b(v_{zz} - v_z))^2} = 0, \quad a, b \neq 0.$$ 

One family of solutions of this equation is very easy to find

$v_z = (\sqrt{(\sigma^2/2a)} - 1)/b$ consequently the corresponding solution $u(s, t)$

$$u(S, t) = \frac{1}{\rho \omega} \left( \sqrt{\frac{\sigma^2}{2a}} - 1 \right) (\log S + at) + c, \quad a > 0,$$

where $c$ is an arbitrary constant.
Properties of the ordinary differential equation

To find other families of solutions we introduce a new dependent variable $y(z) = v_z(z)$ and solve the equation

$$yy_z^2 - 2\left(y^2 + \frac{1}{b}y - \frac{q}{2b^2}\right)y_z + \left(y^2 + \frac{2}{b}y + \frac{1 - q}{b^2}\right)y = 0, \quad a, b \neq 0,$$

where $q = \frac{\sigma^2}{2a} \in R$. This equation can possess exceptional solutions which are the solutions of the system

$$\frac{\partial F(y_z, y)}{\partial y_z} = 0, \quad F(y_z, y) = 0.$$

It will be the case if $a = \sigma^2/8$, i.e. $y(z) = \frac{1}{b}$. 

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The Lipschitz condition for equations of the type $y_z = f(y)$ are satisfied in all points where the derivative $\frac{\partial f}{\partial y}$ exists and is bounded. It is easy to see that this condition will not be satisfied by

$$y = 0, \quad y(z) = \frac{\sigma^2}{8ab}, \quad y = \infty.$$ 

It means that on these lines the uniqueness of solutions of the equation can be lost.
Domain of definition for the $y_z(z)$

If we assume now that $z, y, y_z$ are complex values and denote

$$y(z) = \zeta, \quad y_z(z) = w, \quad \zeta, w \in C,$$

then the differential equation takes the form

$$F(\zeta, w) = \zeta w^2 - 2 \left( \zeta^2 + \frac{1}{b} \zeta - \frac{q}{2b^2} \right) w + \left( \zeta^2 + \frac{2}{b} \zeta + \frac{1 - q}{b^2} \right) \zeta = 0,$$

where $b, q \neq 0$. The polynomial $F(\zeta, w)$ will be irreducible if at all roots $p = w(z)$ either the partial derivative $F_\zeta(\zeta, w)$ or $F_w(\zeta, w)$ are not equal to zero at $p$.

We can interpret the equation above as a relation which defines a Riemann surface $\Gamma : F(\zeta, w) = 0$ for $w = w(\zeta)$ as a compact manifold over the $\zeta$ sphere.

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The number of sheets is 2 except at two branch points \( \zeta_1 = q/(4b), \quad \zeta_2 = \infty \).

Any solution of an irreducible algebraic equation is meromorphic on this compact Riemann surface \( \Gamma \) of genus 0.

This algebraic function \( w(\zeta) \) is uniquely extended analytically over a Riemann surface of exactly 2 sheets on the \( \zeta \) sphere. It can have singular points over the branch points and over the point \( \zeta = 0 \) only.

It means that the meromorphic function \( w(\zeta) \) can not be defined on a manifold of less than 2 sheets over the \( \zeta \) sphere.

To solve the differential equation for the function \( y_\zeta(z) = w(\zeta) \) from this point of view is equivalent to integrate on \( \Gamma \) a differential of the type \( \frac{d\zeta}{w(\zeta)} \) and then to solve the problem similar to the Abel's inverse problem

\[
\int \frac{d\zeta}{w(\zeta)} = z + \text{const.}
\]
The solution of the first order differential equation

The integration can be done very easily because we can introduce an uniformizing parameter \( p \) on the Riemann surface \( \Gamma \) and represent the integral in terms of a rational functions of \( p \) possibly with logarithmic terms.

\[
\left(1 - \frac{4b}{q} \zeta\right) = p^2 q^2
\]

The integration procedure of the equation gives rise to the following relations

\[
2q \log (p - 1) + (q - \sqrt{q} - 2) \log ((p + 1)\sqrt{q} - 2) \\
+ (q + \sqrt{q} - 2) \log ((p + 1)\sqrt{q} + 2) = 2(q - 1)z + c, \quad q \neq 1, q > 0
\]

\[
\frac{1}{1 - p} + \frac{1}{4} \log \frac{(p + 3)^3}{(p - 1)^5} = z + c, \quad q = 1.
\]

\[
2\sqrt{(-q)} \arctan ((p + 1)\sqrt{(-q)/2}) - 2q \log (p - 1) \\
+ (2 - q) \log (4 - q(p + 1)^2) = 2(1 - q)z + c, \quad q < 0.
\]
**Special cases**

For a special value of the parameter $q$ we can invert the equations. Let $q = 4$ then the previous equations take the form

$$(p - 1)^2(p + 2) = c \exp \left(\frac{3z}{2}\right)$$

$$(p + 1)^2(p - 2) = c \exp \left(\frac{3z}{2}\right),$$

where $c$ is an arbitrary constant. The symmetry

$$p \rightarrow -p, \ c \rightarrow -c.$$  

coming from the symmetry of the defining Riemann surface $\Gamma$ and corresponds to change of the sheets on $\Gamma$. Because of this symmetry it is sufficient to study one of the equations for $c \in \mathbb{R}$ or each of these equations for $c > 0$. We will study the second equations for $c \in \mathbb{R}$ and obtain on this way complete class of explicit solutions for the system of equations.

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Plot of different families of invariant solutions for the case

$$\lambda = \omega S.$$
Explicit similarity solutions for the case $\lambda = \omega S$.

$d \in \mathbb{R}, c > 0$

$$u_r(S, t) = -\frac{1}{\omega \rho} \left( 1 + c S^2 e^{\frac{3\sigma^2}{16} t} + \sqrt{2 c S^2 e^{\frac{3\sigma^2}{16} t} + c^2 S^3 e^{\frac{3\sigma^2}{8} t}} \right)^{-\frac{2}{3}}$$

$$-\frac{1}{\omega \rho} \left( 1 + c S^2 e^{\frac{3\sigma^2}{16} t} + \sqrt{2 c S^2 e^{\frac{3\sigma^2}{16} t} + c^2 S^3 e^{\frac{3\sigma^2}{8} t}} \right)^{\frac{2}{3}}$$

$$-\frac{2}{\omega \rho} \log \left( 1 + c S^2 e^{\frac{3\sigma^2}{16} t} + \sqrt{2 c S^2 e^{\frac{3\sigma^2}{16} t} + c^2 S^3 e^{\frac{3\sigma^2}{8} t}} \right)^{-\frac{1}{3}}$$

$$+ \left( 1 + c S^2 e^{\frac{3\sigma^2}{16} t} + \sqrt{2 c S^2 e^{\frac{3\sigma^2}{16} t} + c^2 S^3 e^{\frac{3\sigma^2}{8} t}} \right)^{\frac{1}{3}} - 2 \right) + d$$
Similarity solutions

In case $c < 0$ we can obtain correspondingly three solutions if

$$0 < S \leq |c|^{-\frac{4}{3}} \exp\left(-\frac{\sigma^2}{8} t\right).$$

The first solution is represented by

$$u_1(S, t) = \frac{1}{\omega \rho} \left( \log S + \frac{\sigma^2}{8} t \right) - \frac{2}{\omega \rho} \cos \left( \frac{2}{3} \arccos \left( 1 - |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right)$$

$$- \frac{4}{3\omega \rho} \log \left( 1 + 2 \cos \left( \frac{1}{3} \arccos \left( 1 - |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right) \right)$$

$$- \frac{16}{3\omega \rho} \log \left( \sin \left( \frac{1}{6} \arccos \left( 1 - |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right) \right) + d,$$

where $d \in \mathbb{R}$, $c < 0$. 

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The second solution is given by the formula

\[ u_2(S, t) = \frac{1}{\omega \rho} \left( \log S + \frac{\sigma^2}{8} t \right) - \frac{2}{\omega \rho} \cos \left( \frac{2}{3} \pi + \frac{2}{3} \arccos \left( -1 + |c| S^\frac{3}{2} e^{\frac{3\sigma^2}{16} t} \right) \right) \]

\[ - \frac{4}{3\omega \rho} \log \left( 1 + 2 \cos \left( \frac{1}{3} \pi + \frac{1}{3} \arccos \left( -1 + |c| S^\frac{3}{2} e^{\frac{3\sigma^2}{16} t} \right) \right) \right) \]

\[ - \frac{16}{3\omega \rho} \log \left( \sin \left( \frac{1}{6} \pi + \frac{1}{6} \arccos \left( -1 + |c| S^\frac{3}{2} e^{\frac{3\sigma^2}{16} t} \right) \right) \right) + d. \]

where \( d \in \mathbb{R}, c < 0 \). The first and second solutions are defined for the variables under conditions \( 0 < S \leq |c|^{-\frac{4}{3}} \exp \left( -\frac{\sigma^2}{8} t \right) \). They coincide along the curve

\[ S = |c|^{-4/3} \exp \left( -\frac{\sigma^2}{8} t \right) \]

and cannot be continued further.
The third solution is defined by

\[ u_{3,1}(S, t) = \frac{1}{\omega \rho} \left( \log S + \frac{\sigma^2}{8} t \right) - \frac{2}{\omega \rho} \cos \left( \frac{2}{3} \arccos \left( -1 + \left| c \right| S^2 e^{3\sigma^2 t} \right) \right) \]
\[ - \frac{4}{3\omega \rho} \log \left( -1 + 2 \cos \left( \frac{1}{3} \arccos \left( -1 + \left| c \right| S^2 e^{3\sigma^2 t} \right) \right) \right) \]
\[ - \frac{16}{3\omega \rho} \log \left( \cos \left( \frac{1}{6} \arccos \left( -1 + \left| c \right| S^2 e^{3\sigma^2 t} \right) \right) \right) + d, \]

where \( d \in \mathbb{R} \) and \( S, t \) satisfied the condition

\[ 0 < S \leq |c|^{-\frac{4}{3}} \exp \left( -\frac{\sigma^2}{8} t \right). \]
In case $\log S + \frac{\sigma^2}{8} t > -\frac{4}{3} \ln |c|$ the third solution can be represented by the formula

$$u_{3,2}(S, t) = \frac{1}{\omega \rho} \left( \log S + \frac{\sigma^2}{8} t \right) - \frac{2}{\omega \rho} \cosh \left( \frac{2}{3} \arccosh \left( -1 + |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right)$$

$$- \frac{16}{3 \omega \rho} \log \left( \cosh \left( \frac{1}{6} \arccosh \left( -1 + |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right) \right)$$

$$- \frac{4}{3 \omega \rho} \log \left( -1 + 2 \cosh \left( \frac{1}{3} \arccosh \left( -1 + |c| S^{\frac{3}{2}} e^{\frac{3\sigma^2}{16} t} \right) \right) \right) + d.$$
Theorem

The equation $\Delta = 0$ possesses non-trivial invariant solutions for the special form of the function $\lambda(S) = \omega S^k$ only.

In case $\lambda(S) = \omega S^k$ the invariant solutions of the equation are defined by ordinary differential equations. In special cases $k = 0, 1$ ODEs are of an autonomous type.

If $\lambda(S) = \omega S^k$, i.e. $k = 1$, then the invariant solutions of the equation can be defined by the set of the first order ordinary differential equations.

If additionally the parameter $q = 4$, or equivalent in the first invariant we chose $a = \sigma^2/8$ then the complete set of invariant solutions of the eqn. can be found exactly. This set of invariant solutions is given by formulas (7)–(11) and by solutions

$$u(S, t) = d, \quad u(S, t) = -3/b \left( \log S + \sigma^2 t/8 \right), \quad u(S, t) = 1/b \left( \log S + \sigma^2 t/8 \right),$$

where $d$ is an arbitrary constant. This set of invariant solutions is unique up to the transformations of the symmetry group $G_{\Delta}$ given by our theorem.
Properties of invariant solutions

Using the exact formulas for solutions we retain the first two terms and obtain as $S \to 0$

\[ u_1(S, t) \sim -\frac{1}{b} \left( 2 + \frac{4}{3} \log (|c|^2 2^{-2} 3^{-3}) + \frac{3}{8} \sigma^2 t + 3 \log S + \mathcal{O}(S^{3/2}) \right), \quad (11) \]

\[ u_2(S, t) \sim \frac{1}{b} \left( 1 + \frac{1}{3} \log (2^8 3^{-6} |c|^{-2}) + \frac{2^3 \sqrt{2} |c|}{3^{4/3}} e^{\frac{3}{32} \sigma^2 t} S^{3/4} + \mathcal{O}(S^{3/2}) \right), \quad (12) \]

\[ u_{3,1}(S, t) \sim \frac{1}{b} \left( 1 + \frac{1}{3} \log (2^8 3^{-6} |c|^{-2}) - \frac{2^3 \sqrt{2} |c|}{3^{4/3}} e^{\frac{3}{32} \sigma^2 t} S^{3/4} + \mathcal{O}(S^{3/2}) \right), \quad (13) \]

\[ u_r(S, t) \sim -\frac{1}{b} \left( 2 + 2 \log (2 3^{-2} c) + \frac{3}{8} \sigma^2 t + 3 \log S + \mathcal{O}(S^{5/4}) \right). \quad (14) \]
If $S$ is large enough we have just two solutions. The asymptotic behavior both solutions $u_r(S,t), u_{3,2}(11)$, coincides in the main terms as $S \to \infty$ and is given by formula

$$u_r(S,t), u_{3,2}(S,t) \sim -\frac{1}{b} \left( (2 |c|)^{2/3} e^{\frac{3\sigma^2}{8} t} S + \log S + O(1) \right), \quad S \to \infty. \quad (15)$$

The main terms in formulas (12)-(16) depends on the time and on the constant $c$. 

*PDE&Finance, Stockholm, 2007*
The reduction to the linear case

The equation

$$\Delta(u) = u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{1 - \rho S \lambda(S) u_{SS}^2} = 0,$$

by $\rho \to 0$ reduces to the linear Black-Scholes equation. The solution which we obtained now will be complete blow up by $\rho \to 0$ because of the last term in the formula.

The solution $u(S, t)$ has no one counterpart in the linear case.
Plot of the solution \( u_r(S, t) \) for the parameters \(|c| = 0.5\), \( b = 1.0\), \( \sigma = 0.3\), \( d1 = 11.5\), \( d2 = -9.0\). The variables \( S, t \) lie in intervals \( S \in (0.04, 3.7) \) and \( t = 0.\) (doted line), \( t = 5.\) (short dashed line) and \( t = 10.\) (long dashed line). Payoff for a long strip with 60 Puts and 10 Calls with exercise prise 0.2 marked by thin solid line.
Plot of different families of invariant solutions, $k = 0$
Plot of different families of invariant solutions, $k = 0$
The model of Musiela & Zariphopoulou 2004

Two risky assets: a stock that can be traded and a nontraded asset on which a European claim is written. Assets are modeled as diffusion processes denoted by $S$ and $Y$. The stock price is a lognormal diffusion satisfying

$$dS_s = \mu S_s ds + \sigma S_s dW^1_s, \quad t < s,$$

with $S_s = S > 0$, and the level of the nontraded asset is given by

$$dY_s = b(Y_s, s) ds + a(Y_s, s) dW_s, \quad t < s,$$

with $Y_t = y \in \mathbb{R}$. 
Theorem (Musiel & Zariphopoulou 2004).

Assume exponential preferences and that the dynamics of the traded and nontraded asset are given by previous equations. Then, the writers indifference price of a European claim $u(y, t)$ satisfies the quasilinear equation

$$u_t + \frac{\sigma^2(y, t)}{2} u_{yy} + \left( b(y, t) - \rho \frac{\mu}{\sigma} a(y, t) \right) u_y + \frac{\gamma}{2} (1 - \rho^2) a^2(y, t) u_y^2 = 0.$$
Invariant solutions of the Musiela & Zariphopoulou model

Let \( \phi(y, t) = \log y - \kappa^2 t \), where \( \kappa \) is a constant, the functions \( a(y, t) \) and \( b(y, t) \) have special forms

\[
a(y, t) = \kappa y g(\phi), \quad b(y, t) = \kappa y \left( \frac{\mu}{\sigma} g(\phi) - \kappa f(\phi) \right),
\]

where \( g(\phi), f(\phi) \) are arbitrary functions and \( g(\phi) \neq 0 \). Then

\[
u(y, t) = -\frac{1}{1 - \rho^2} \log \left( \int e^{\Phi(\phi)} d\phi \right),
\]

where

\[
\Phi(\phi) = \int \frac{g^2(\phi) d\phi}{(1 + g^2(\phi) + f(\phi))}.
\]