No-arbitrage bounds on implied volatility

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The price of a European call option with strike $K$ and maturity $T$

$$C_t(K, T) = S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

- $S_t$ is the price of the underlying stock
- $r$ is the spot interest rate

$$d_{1,2} = \frac{\log \left( \frac{S_t}{e^{-r(T-t)}K} \right)}{\sqrt{T - t} \sigma} \pm \frac{\sqrt{T - t} \sigma}{2},$$

- $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2} dt$
Introduction
The term structure approach
The smile
Impossibility theorems

The Black–Scholes formula
Assumptions and notation

$\sigma$ is the volatility of the underlying stock
Unlike other parameters, not directly observed

$$\sigma^2 = \frac{\langle \log S \rangle_T}{T} = \frac{\text{Var}(\log S_T)}{T}$$

Liquid options priced by market already
Options often quoted in terms of implied volatility
The assumptions

- No arbitrage
- Calls of all maturities and strikes liquidly traded
- Zero interest rate

Let \((S_t)_{t \geq 0}\) be a non-negative martingale with \(S_0 = 1\).

Price of European call option with strike \(K\) and maturity \(T\)

\[
C_t(K, T) = \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t]
\]
Define the function $F_{BS} : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, 1)$ via

$$F_{BS}(k, \nu) = \begin{cases} 
    \Phi\left( -\frac{k}{\sqrt{\nu}} + \frac{\sqrt{\nu}}{2} \right) - e^k \Phi\left( -\frac{k}{\sqrt{\nu}} - \frac{\sqrt{\nu}}{2} \right) & \text{if } \nu > 0 \\
    (1 - e^k)^+ & \text{if } \nu = 0
\end{cases}$$

Definition

The non-negative random variable $\Sigma_t(k, \tau)$ is defined by

$$\mathbb{E}\left[ (\frac{S_{t+\tau}}{S_t} - e^k)^+ | \mathcal{F}_t \right] = F_{BS}(k, \tau \Sigma_t(k, \tau)^2)$$
Question

*How does the no-arbitrage assumption contrain the dynamics of the random field*

\[ (\Sigma_t(k, \tau))_{t \geq 0, k \in \mathbb{R}, \tau > 0} \]
Note the analogy with interest rate theory: If 

\[(r_t)_{t \geq 0}\]

is the spot interest rate, the forward rate \(f_t(\tau)\) is defined by

\[
e^{-\int_0^\tau f_t(s) \, ds} = \mathbb{E}[e^{-\int_{t+\tau}^{t+\tau} r_s \, ds} | \mathcal{F}_t]
\]

Note

\[f_t(0) = r_t\]
The Heath-Jarrow-Morton (1992) drift condition: If

\[ df_t(\tau) = a_t(\tau)dt + b_t(\tau)dW_t \]

then

\[ a_t(\tau) = \frac{\partial}{\partial \tau} f_t(\tau) + b_t(\tau) \int_0^\tau b_t(s)ds \]
A HJM approach to implied volatility: If

$$dS_t = S_t \sigma_t dW_t$$

for some process $$(\sigma_t)_{t \geq 0}$$ then

$$\Sigma_t(0, 0) = \sigma_t$$

If

$$d\Sigma_t(k, \tau) = a_t(k, \tau) dt + b_t(k, \tau) dW_t$$

then

$$a_t(k, \tau) = \frac{\partial}{\partial \tau} \Sigma_t(k, \tau) + \text{A MESS}$$
▶ Dupire (1994)
▶ Derman and Kani (1997)
▶ Schönbucher (1998)
▶ Cont and Da Fonseca (2001)
▶ Balland (2002)
▶ Durrleman (2004, 2007)
▶ Schwiezer and Wissel (2005, 2007)
▶ Jacod and Protter (2006)
▶ Carmona and Nadtochiy (2007)
Theorem (Dybvig–Ingersoll–Ross (1996))

Let

\[ \limsup_{\tau \uparrow \infty} f_t(\tau) = \ell_t. \]

Then

\[ \ell_t \geq \ell_s \]

for \( t \geq s \geq 0 \).

Parallel shifts of the term structure:

**Proposition**

Suppose

\[ f_t(\tau) = f_0(\tau) + \xi_t \]

for some process \((\xi_t)_{t \geq 0}\). If

- \( r_t \geq 0 \) almost surely
- \( \sup_{t \geq 0} \mathbb{E}(r_t) < \infty \)

then \( \xi_t = 0 \) for all \( t \geq 0 \).
Parallel shifts of the implied volatility surface:

**Conjecture (Ross)**

*Suppose*

\[ \Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t \]

*for some process \((\xi_t)_t \geq 0\).*  *Then \(\xi_t = 0\) for all \(t \geq 0\).*
Theorem (Lee (2004))

Let

$$\beta = \limsup_{k \uparrow \infty} \frac{\tau \sum(k, \tau)^2}{k}.$$

Then

$$\frac{1}{2\beta} + \frac{\beta}{8} + \frac{1}{2} = \sup \{p > 1 : \mathbb{E}(S^p_\tau) < \infty \}.$$

A similar formula holds as $k \downarrow -\infty$.

See Benaim and Friz (2006) for more precise asymptotics.
Proposition

\[ \lim_{k \downarrow -\infty} \sqrt{\tau} \Sigma(k, \tau) - \sqrt{-2k} = \Phi^{-1}(\mathbb{P}(S_\tau = 0)) \]

For example, if \( dS_t = \alpha S_t^\beta dW_t \) with \( 0 < \beta < 1 \) then

\[ \sqrt{\tau} \Sigma(k, \tau) = \sqrt{-2k} - \Phi^{-1} \circ \Gamma\left(\frac{1}{2(1-\beta)}, \frac{1}{2(1-\beta)^2\alpha^2\tau}\right) + o(1) \]

where \( \Gamma(\gamma, x) = \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} e^{-t} dt \).
Proposition

If $S_\tau > 0$ a.s. then

$$\Sigma(-k, \tau) = \hat{\Sigma}(k, \tau)$$

where

$$\mathbb{E}[(\hat{S}_\tau - e^k)^+] = F_{BS}(k, \tau \hat{\Sigma}(k, \tau)^2)$$

and

$$\mathbb{P}(\hat{S}_\tau \leq t) = \mathbb{E}(S_\tau \mathbb{1}_{\{S_\tau < t\}}).$$
For example, if

\[
    dS_t = S_t \sigma(Y_t) dW_t \\
    dY_t = \alpha(Y_t) dt + \beta(Y_t) dW_t + \gamma(Y_t) dW_t^\perp
\]

then

\[
    d\hat{S}_t = \hat{S}_t \sigma(\hat{Y}_t) dW_t \\
    d\hat{Y}_t = (\alpha(\hat{Y}_t) + \beta(\hat{Y}_t) \sigma(\hat{Y}_t)^2) dt + \beta(\hat{Y}_t) dW_t + \gamma(\hat{Y}_t) dW_t^\perp.
\]

Renault-Touzi’s (1996) result follows from this.
Proposition

The smile is symmetric

\[ \Sigma(k, \tau) = \Sigma(-k, \tau) \]

if and only if

\[ \mathbb{E}(S^p_\tau) = \mathbb{E}(S^{1-p}_\tau) \]

for all \( 0 \leq p \leq 1 \).
Theorem

Assume that the law of $S_{\tau}$ is continuous for all $\tau > 0$.

- $\partial_k \Sigma(k, \tau)^2 < \frac{4}{\tau}$ for all $k \geq 0$

- $\partial_k \Sigma(k, \tau)^2 > -\frac{4}{\tau}$ for all $k \leq 0$

If $S_t \to 0$ a.s., then

$$\limsup_{\tau \to \infty} \sup_{k \in [-M, M]} \tau |\partial_k \Sigma(k, \tau)^2| \leq 4$$

This sharpens the result of Carr and Wu (2002).
The inequality is sharp in the sense that there exists a martingale \((S_t)_{t \geq 0}\) such that

\[ \tau \partial_k \sum(k, \tau)^2 \to -4 \]

as \(\tau \uparrow \infty\) uniformly for \(k \in [-M, M]\).
For comparison, a consequence of Lee’s results:

**Proposition**

Assume that the law of $S_\tau$ is continuous for all $\tau > 0$.

- There exists a $k_+ \geq 0$ such that
  \[
  \partial_k \Sigma(k, \tau)^2 < \frac{2}{\tau} \text{ for all } k \geq k_+
  \]

- If $S_\tau > 0$ a.s. then there exists a $k_- \leq 0$ such that
  \[
  \partial_k \Sigma(k, \tau)^2 > -\frac{2}{\tau} \text{ for all } k \leq k_-
  \]
The condition $S_t \to 0$ almost surely is natural since the following are equivalent:

- $S_t \to 0$ almost surely.
- There exists a $k \in \mathbb{R}$ such that $\tau \Sigma(k, \tau)^2 \uparrow \infty$
- $\tau \Sigma(k, \tau)^2 \uparrow \infty$ for all $k \in \mathbb{R}$.
- $\mathbb{E}[(S_\tau - K)^+] \uparrow S_0$ for all $K > 0$.
- There exists a $K > 0$ such that $\mathbb{E}[(S_\tau - K)^+] \uparrow S_0$
Proposition

\[
\lim_{\tau \uparrow \infty} \sum(k, \tau) - \left( -8 \log \frac{\mathbb{E}(S_\tau \wedge 1)}{\tau} \right)^{1/2} = 0
\]

for all \( k \in \mathbb{R} \).

See Lewis (2000) and Jacquier (2006, 2007) for detailed \( T \uparrow \infty \) asymptotics for some popular models.
Theorem

For any $k_1, k_2 \in \mathbb{R}$ we have

$$\limsup_{\tau \uparrow \infty} \Sigma_t(k_1, \tau) - \Sigma_s(k_2, \tau) \geq 0$$

for $t \geq s \geq 0$. There exist examples for which the inequality is strict.
Theorem

Suppose

\[ \Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t \]

for some process \((\xi_t)_{t \geq 0}\). Then \(\xi_t \geq 0\).

Furthermore, let

\[ g_p(t) = \frac{1}{p(p-1)} \log \mathbb{E}(S^p_t). \]

If there exists a \(p \in \mathbb{R}\) and a \(\tau > 0\) such that

\[ g_p(t + \tau) \leq g_p(t) + g_p(\tau) \]

then \(\xi_t = 0\).
If $\xi_t = 0$ for all $t \geq 0$ and $S_t \to 1$ in probability as $t \downarrow 0$ then $(S_t)_{t \geq 0}$ is an exponential Levy process.
The conjecture is false for implied average variance $\Sigma_t(k, \tau)^2$.

**Proposition**

The martingale

$$S_t = e^{-t^4/2+W_t^2}$$

has the property that

$$\Sigma_t(k, \tau)^2 = \Sigma_0(k, \tau)^2 + \xi_t$$

where

$$\Sigma_0(k, \tau) = \sqrt{\tau}$$

and

$$\xi_t = 2t$$
Proposition

Suppose

\[ \Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t \]

for some process \((\xi_t)_{t \geq 0}\). If

\[ S_t = e^{-F(t)/2}W_{F(t)} \]

for a positive increasing some function \(F\) with \(F(0) = 0\). Then

\[ \xi_t = 0. \]