ON THE SPECTRUM AND EIGENFUNCTIONS
OF THE SCHRÖDINGER OPERATOR
WITH AHARONOV-BOHM MAGNETIC FIELD

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Abstract. We explicitly compute the spectrum and eigenfunctions of the magnetic Schrödinger operator
\[ H(\tilde{A}, V) = (i\nabla + \tilde{A})^2 + V \]
in \( L^2(\mathbb{R}^2) \), with Aharonov-Bohm vector potential,
\[ \tilde{A}(x_1, x_2) = a(-x_2, x_1)/|x|^2, \]
and either quadratic or Coulomb scalar potential \( V \). We also determine sharp constants in the CLR inequality, both dependent on the fractional part of \( a \) and both greater than unity. In the case of quadratic potential, it turns out that the LT inequality holds for all \( \gamma \geq 1 \) with the classical constant, as expected from the non-magnetic system (harmonic oscillator).

1. Introduction

The main aim of this article is to determine explicit constants in the Lieb-Thirring (LT) and Cwikel-Lieb-Rozenblyum (CLR) inequalities for a class of exactly solvable quantum-mechanical models. We consider the magnetic Schrödinger operator
\[ H(\tilde{A}, V) = (i\nabla + \tilde{A})^2 + V, \]
in \( L^2(\mathbb{R}^2) \) with Aharonov-Bohm vector potential,
\[ \tilde{A}(x_1, x_2) = a(-x_2, x_1)/|x|^2, \quad a \in \mathbb{R} \setminus \mathbb{Z}, \]
and with two different choices of scalar potential. In both cases the optimal CLR constant depends on \( |a - m_1| \), where \( m_1 \) is the best integer approximation of \( a \).

We initially use a quadratic scalar potential, \( V(x_1, x_2) = \beta|x|^2 \), where \( \beta \in \mathbb{R}^+ = (0, \infty) \). The operator is then unitarily equivalent to the two-dimensional harmonic oscillator if the magnitude \( \alpha \) is an integer. Such an operator has already been considered, for instance in [2] and [6]. In the latter work, the authors construct a solution of the time-dependent Schrödinger equation. In the corresponding classical system, whose trajectories are given by Hamilton’s equation, the particles move in periodic orbits around the singularity, unaffected by the Aharonov-Bohm field. Quantum-mechanically, however, the effect of the magnetic field can be observed in the solutions of the Schrödinger equation. It turns out that the spectrum and eigenfunctions of the operator (1.1) can be computed explicitly (Theorem 1). Here again, one sees a contribution of the Aharonov-Bohm effect in so far as the eigenfunctions differ from those of the harmonic oscillator when the magnitude \( \alpha \) is non-integer.

We moreover prove that the LT inequality, i.e.,
\[ \text{Tr}(H(\tilde{A}, V) - \lambda)^\gamma \leq \frac{R_\gamma}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (a(x, \xi) - \lambda)^\gamma \, dx \, d\xi, \]
holds true for the operator with the classical constant \( R_\gamma = 1 \) for all \( \gamma \geq 1 \) (Theorem 3). Such a result could not have been deduced from the results in [3] or [5], where the authors consider non-magnetic Schrödinger operators. It is known that

I am grateful to my advisor Ari Laptev for providing me with this problem to study. I also want to thank the ESF research programme Spectral Theory and Partial Differential Equations (SPECT) for support and inspiration.
non-magnetic systems cannot satisfy the CLR inequality \( (\gamma = 0) \) in two dimensions.

With the Aharonov-Bohm field, however, this inequality is sharp with

\[
R_0 = \begin{cases} 
\frac{(1+|\alpha-m_1|)^2}{2(1-\frac{1}{2}|\alpha-m_1|)^2} & \text{if } 0 < |\alpha - m_1| \leq 3\sqrt{2} - 4, \\
\frac{1}{(1-\frac{1}{2}|\alpha-m_1|)^2} & \text{if } 2\sqrt{2} - 4 \leq |\alpha - m_1| \leq \frac{1}{2},
\end{cases}
\]

which is always greater than unity (Theorem 2).

Parallel results are obtained in the second part for the Coulomb potential,

\[ V(x_1, x_2) = -\beta/|x|. \]

Unlike the quadratic potential it is not confining, and consequently the point spectrum is entirely negative (Theorem 4). The LT inequality is trivial if \( \gamma \geq 1 \) and we establish (Theorem 5) that the sharp CLR constant is

\[
R_0 = \begin{cases} 
\frac{1}{(1+|\alpha-m_1|)^2} & \text{if } 0 < |\alpha - m_1| \leq 2\sqrt{2} - \frac{5}{2}, \\
\frac{1}{2(1-\frac{1}{2}|\alpha-m_1|)^2} & \text{if } 2\sqrt{2} - \frac{5}{2} \leq |\alpha - m_1| \leq \frac{1}{2},
\end{cases}
\]

going again \( R_0 > 1 \) for all \( \alpha. \)

**Part 1. Quadratic scalar potential**

2. Spectrum and eigenfunctions

In this section we will see that the eigenvalue problem for \( H(\vec{A}, V) \) with quadratic potential can be reduced to Whittaker’s differential equation. The spectrum of the operator turns out to have a close connection with that of the harmonic oscillator.

2.1. Separation of variables. We may use the decomposition

\[
L^2(\mathbb{R}^2) = L^2(\mathbb{R}^+, r dr) \otimes L^2(S^1) = \bigoplus_{m \in \mathbb{Z}} \left( L^2(\mathbb{R}^+, r dr) \otimes [e^{im\theta}/\sqrt{2\pi}] \right),
\]

(2.1)

where \([\cdot]\) denotes the linear span, to express the Aharonov-Bohm operator as

\[
H(\vec{A}, V) = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( i \frac{\partial}{\partial \theta} + \alpha \right)^2 + \beta r^2 = \bigoplus_{m \in \mathbb{Z}} (H_m \otimes I_m),
\]

(2.2)

where \( I_m \) is the identity on \([e^{im\theta}/\sqrt{2\pi}]\) and

\[
H_m = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} (\alpha - m)^2 + \beta r^2.
\]

To remove the weight \( r \) we introduce the unitary mapping

\[
U : L^2(\mathbb{R}^+, r dr) \to L^2(\mathbb{R}^+, dr)
\]

\[
f(r) \mapsto \sqrt{r} f(r),
\]

(2.3)

which transforms \( H_m \) into

\[
\tilde{H}_m = U H_m U^{-1} = -\frac{d^2}{dr^2} + (\alpha - m)^2 - \frac{1}{r^2} + \beta r^2.
\]

(2.4)

Following (2.1) we write

\[
u(r, \theta) = \sum_{m=-\infty}^{\infty} u_m(r) e^{im\theta}
\]

and the corresponding quadratic form decomposes accordingly:

\[
\tilde{a}[u] = \sum_{m=-\infty}^{\infty} \tilde{a}_m[u_m],
\]

(2.5)

where

\[
\tilde{a}_m[u] = \int_0^\infty \left( \frac{|du|^2}{dr} + (\alpha - m)^2 - \frac{1}{r^2} |u|^2 + \beta r^2 |u|^2 \right) dr.
\]
The operator $H(\Lambda, V)$ will be considered as the Friedrichs extension of the differential expression (2.2) on $C^\infty_0(\mathbb{R}^2 \setminus \{0\})$. By an application of the classical Hardy inequality (and a standard density argument), one can prove that its domain consists of all $H_0^1$ functions such that the quadratic form (2.5) is finite.

2.2. Eigenfunctions. The spectrum of this operator is discrete and can be calculated explicitly. Our goal is to find all eigenfunctions of $H(\Lambda, V)$, i.e., all $\varphi_m e^{im\theta}$ which are eigenfunctions of $H_m \otimes I_m$. Taking into account the mapping (2.3) we have

$$H_m \varphi_m = E \varphi_m \iff \tilde{H}_m \tilde{\varphi}_m = E \tilde{\varphi}_m,$$

where $\tilde{\varphi}_m = U \varphi_m$. Substituting further

$$\tilde{\varphi}_m(r) = \tilde{\varphi}_m(r^2) \sqrt{\rho},$$

in (2.6), we obtain the equation

$$4\rho^2 \left[ \tilde{\varphi}_m''(r^2) + \left( -\frac{\beta}{4} + \frac{E/4}{\rho^2} + \frac{1 - (a-m)^2}{\rho^4} \right) \tilde{\varphi}_m(r^2) \right] = 0$$

$$\iff 4\rho^2 \left[ \tilde{\varphi}_m''(\sqrt{\rho}r^2) + \left( -\frac{1}{4} + \frac{\lambda}{z} + \frac{1 - \mu^2}{z^2} \right) \tilde{\varphi}_m(\sqrt{\rho}r^2) \right] = 0.$$
For large $z$ the Whittaker functions have the following asymptotics: \[ M_{\lambda, \mu}(z) = \left( e^{i\pi \lambda} \Gamma(2\mu + 1) \right) \left( \frac{e^{\pi \gamma}}{\Gamma(\mu - \lambda + \frac{1}{2})} (z)^{-\lambda} e^{-z^2/2} + \frac{e^{\pi \gamma}}{\Gamma(\mu + \lambda + \frac{1}{2})} (z)^{\lambda} e^{-z^2/2} \right) + \mathcal{O}(z^{-1}). \]

We deduce that
\[ \varphi_m^{+}(r) = \left( \frac{e^{i\pi \beta}}{\Gamma(\alpha - m + m)} \left( (\alpha - m) \right)^{-\lambda} e^{-r^2/2} + \frac{e^{i\pi \beta}}{\Gamma(\alpha + m + m)} \left( (\alpha + m) \right)^{\lambda} e^{-r^2/2} \right) + \mathcal{O}(r^{-1}). \]

Theorem 1. The $L^2(\mathbb{R}^2)$ eigenfunctions of the operator (1.1) with
\[ \tilde{A}(x_1, x_2) = \alpha(-x_2, x_1)/|x|^2 \quad \text{and} \quad V(x_1, x_2) = \beta|x|^2, \]
where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\beta \in \mathbb{R}_+$, are
\[ e^{i\pi \beta} M_{E(m,n)/4\pi, \frac{1}{2}} \left( \sqrt{\beta} e^{-2z} \right), \]
where $m \in \mathbb{Z}$ and $M_{\lambda, \mu}$ is defined in (2.7). The eigenvalues are
\[ E(m,n) = 2\sqrt{\beta}(1 + |\alpha - m| + 2n), \quad n \in \mathbb{N}. \]

The multiplicity of a given eigenvalue equals how many times it appears as $m$ runs over $\mathbb{Z}$ and $n$ over $\mathbb{N}$.

2.3. Eigenvalues. For future convenience we shall write the eigenvalues as two increasing sequences:
\[ E_j, \pi = \epsilon_j + 2\sqrt{\beta} \pi, \quad j = 1, 2, \pi \in \mathbb{N}. \]

Here $\epsilon_j$ denotes the lowest eigenvalues,
\[ \epsilon_1 = \min_{m \in \mathbb{Z}} 2\sqrt{\beta}(1 + |\alpha - m|) = 2\sqrt{\beta}(1 + |\alpha - m_1|) \]
\[ \epsilon_2 = \min_{m_1 \neq m \in \mathbb{Z}} 2\sqrt{\beta}(1 + |\alpha - m|) = 6\sqrt{\beta} - \epsilon_1, \]

which coincide if $\alpha$ is a half-integer. In fact,
\[ 1 + |\alpha - m| + 2\pi = \epsilon_j + \epsilon_j + 2\pi = \epsilon_j + \pi \]
and since \( p = m' + 2n \) has \( \lfloor p/2 \rfloor + 1 \) solutions in \( \mathbb{N} \times \mathbb{N} \), the multiplicity of the eigenvalue \( E_{j,p} \) will be \( N(p) = \lfloor p/2 \rfloor + 1 \).

\[ \text{Figure 1. The first eigenvalues, normalised by } \sqrt{\beta} \]

In Figure 1 we have plotted the first eigenvalues. The spectrum has a close connection with that of the two-dimensional harmonic oscillator,

\[ E_{\text{h.o.}}(p) = 2p, \quad N_{\text{h.o.}}(p) = p, \quad p = 1, 2, \ldots. \quad (2.14) \]

The eigenvalues have moved apart from their original positions by a distance which is proportional to the fractional part of \( \alpha \).

3. Eigenvalue inequalities

We now consider the two-dimensional Lieb-Thirring inequality

\[ \text{Tr}(H(\vec{A}, V) - \lambda)^\gamma \leq \frac{R_\gamma}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (a(x, \xi) - \lambda)^\gamma \, dx d\xi, \quad (3.1) \]

which is known to hold for all \( \gamma > 0 \) for the harmonic oscillator in the absence of a magnetic field. In this case the constant \( R_\gamma = 1 \) if \( \gamma \geq 1 \) [3], but as a general fact \( R_\gamma > 1 \) if \( \gamma < 1 \) [4]. In the special case \( \gamma = 0 \) the inequality is usually named for Cwikel, Lieb and Rozenblum. It fails for non-magnetic systems unless the dimension is at least 3.

By unitary equivalence (3.1) holds for the Aharonov-Bohm operator if the magnetic potential has integer magnitude \( \alpha \). We shall address the question whether this is true also in the case of non-integer magnitude. We are led to study the cases \( \gamma = 0 \) and \( \gamma = 1 \) by the above prediction and the well-known result by Aizenmann and Lieb [1]: If \( R_\gamma \) is finite for some \( \gamma \geq 0 \), then \( R_\gamma' \leq R_\gamma \) for all \( \gamma' \geq \gamma \).

3.1. Right-hand side. Let us first calculate the right-hand side of (3.1). The Schrödinger operator \( H(\vec{A}, V) \) is a pseudodifferential operator with symbol

\[ a(x, \xi) = \left( -\xi_1 - \frac{\alpha x_2}{|x|^2}, -\xi_2 + \frac{\alpha x_1}{|x|^2} \right)^2 + \beta |x|^2. \]

By means of the substitution

\[ y_1 = \sqrt{\beta} x_1, \quad y_2 = \sqrt{\beta} x_2, \quad \eta_1 = -\xi_1 - \frac{\alpha x_2}{|x|^2}, \quad \eta_2 = -\xi_2 + \frac{\alpha x_1}{|x|^2} \]

the symbol simplifies to \(|\eta|^2 + |y|^2\). The integral is therefore zero for \( \lambda \leq 0 \). For positive \( \lambda \) we have

\[
\begin{align*}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (a(x, \xi) - \lambda)^\gamma \, dx d\xi &= \frac{1}{\beta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (|y|^2 + |\eta|^2 - \lambda)^\gamma d\eta d\eta \\
&= \frac{1}{\beta} \int_{|y|^2 + |\eta|^2 \leq \lambda} (\lambda - |y|^2 - |\eta|^2)^\gamma d\eta d\eta \\
&= \frac{(2\pi)^2}{\beta} \int_{r^2 + \rho^2 \leq \lambda} (\lambda - r^2 - \rho^2)^\gamma r \rho d\rho \\
&= \frac{(2\pi)^2}{\beta} \int_0^{\sqrt{\lambda}} \cos \psi \sin \psi d\psi \int_0^{\sqrt{\lambda}} (\lambda - R^2)^\gamma R^3 dR \\
&= (2\pi)^2 \frac{\lambda^{\gamma+2}}{4\beta(\gamma + 1)(\gamma + 2)}.
\end{align*}
\]

The result is independent of the magnetic field.
3.2. Left-hand side, case $\gamma = 0$. The left-hand side can be written

$$\text{Tr}(H(A, V) - \lambda)^\gamma = \sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(\lambda - E_{j,p})^\gamma,$$

(3.2)

which, if $\gamma = 0$, is simply the number $N_\lambda$ of eigenvalues (counted with their multiplicities) less than or equal to $\lambda$. For any $\gamma$ we can restrict the computations to the case $\beta = 1$, because $\sum_{j,p} N(p)(\lambda - E_{j,p})^\gamma \leq R_\gamma \lambda^{\gamma+2}/4(\gamma + 1)(\gamma + 2)$ implies that

$$\sum_{j,p} N(p)(\lambda - \sqrt{\beta}E_{j,p})^\gamma = \beta^{\gamma/2} \sum_{j,p} N(p) \left( \frac{\lambda}{\sqrt{\beta}} - E_{j,p} \right)^\gamma \leq \beta^{\gamma/2} \frac{R_\gamma (\lambda/\sqrt{\beta})^{\gamma+2}}{4(\gamma + 1)(\gamma + 2)}.$$

Since $2 < \epsilon_j \leq 3$ irrespectively of $\alpha$, there is exactly one point in the spectrum between two consecutive integers. The sum (3.2) is particularly easy to compute if $\lambda$ is an even integer. Recall that the spectrum begins at 2 and that the interval $[4p - 2, 4p + 2]$ contains four eigenvalue points, each with multiplicity $p$. Thus, if $\lambda = 4n + 2$,

$$N_\lambda = \sum_{p=1}^{n} 4p = 2n(n + 1) = \left( \frac{\lambda}{2} - 1 \right) \left( \frac{1}{2} \left( \frac{\lambda}{2} - 1 \right) + 1 \right) = \frac{\lambda^2}{8} - \frac{1}{2}.$$

Similarly, if $\lambda = 4n$,

$$N_\lambda = \sum_{p=1}^{n} 4p - 2n = 2n^2 = \frac{\lambda^2}{8}.$$

Figure 2. Plots of $\lambda^2/8$ and $N_\lambda$ on the interval $[4n - 2, 4n + 2]$

Figure 2 contains all information needed to determine a lower bound on the constant

$$R_0 = \sup_\lambda \frac{N_\lambda}{\lambda^2/8}.$$
$N_\lambda$ being non-decreasing, this supremum is necessarily attained at some point of the spectrum, where $N_\lambda$ has a jump increase. Formulae (2.12)–(2.13) tell us that, e.g.,

$$\frac{N_{\epsilon_1+4(n-1)}}{(\epsilon_1 + 4(n-1))^2/8} = \frac{8(2n^2 - n)}{2^2(1+2(n-1) + |\alpha - m_1|)^2} = \frac{2n(2n-1)}{(2n-1 + |\alpha - m_1|)^2}.$$  

Hence the interval $[4n-2, 4n+2]$ provides the bound

$$R_0 \geq \max \left\{ \frac{2n(2n-1)}{(2n-1 + |\alpha - m_1|)^2}, \frac{4n^2}{(2n-|\alpha - m_1|)^2}, \frac{2n(2n+1)}{(2n + |\alpha - m_1|)^2}, \frac{4n(n+1)}{(2n + 1 - |\alpha - m_1|)^2} \right\}.$$  

We may view these expressions as functions of $n = 1, 2, 3, \ldots$. They are decreasing if, respectively,

$$n > \frac{|\alpha - m_1| - 1}{2(2|\alpha - m_1| - 1)}; \quad n > 0; \quad n > \frac{|\alpha - m_1|}{2 - 4|\alpha - m_1|}; \quad n > 1 - |\alpha - m_1|.$$  

A somewhat lengthy but altogether elementary examination of all possible cases shows that it is enough to consider $n = 1$, i.e., to solve the maximisation problem on the interval $[2, 6]$. The conclusion is

**Theorem 2.** When $\gamma = 0$, inequality (3.1) is sharp with

$$R_0 = \begin{cases} \frac{2}{1+\alpha - m_1}, & \text{if } 0 < |\alpha - m_1| \leq 3\sqrt{2} - 4, \\ \frac{1}{1-\frac{3\sqrt{2}}{4}}, & \text{if } 3\sqrt{2} - 4 \leq |\alpha - m_1| \leq \frac{1}{2}. \end{cases}$$

Apparently $R_0$ is always greater than or equal to $2(1 + \sqrt{2})^2/9 \approx 1.295$ (which indeed confirms the result in [4]) and $R_0 / 2$ as $\alpha$ approaches an integer. This fact can also be established by direct calculations with the non-magnetic eigenvalues (2.14).

3.3. **Left-hand side, case $\gamma = 1$.** We attempt to show that $R_1 = 1$, as is the case of the harmonic oscillator. Taking $\beta = 1$ as previously, we shall prove that the quantity (3.2) does not exceed $\lambda^3/24$. Since each point in the spectrum stays between the same consecutive integers when $\alpha$ varies, and because $\epsilon_1 + \epsilon_2 = 6$, the sum is independent of $\alpha$ when $\lambda$ is an even integer. We will compute the sum for such $\lambda$ and then use convexity to determine the value of the constant.

Consider first $\lambda = 4n + 2$ and write $[2, 4n+2] = \bigcup_{p=1}^{n} [4p-2, 4p+2]$. The four points in the spectrum located on $[4p-2, 4p+2]$ all have multiplicity $p$. They contribute to the sum in the following way:

$$4(p-1) + \epsilon_1 \quad \text{gives} \quad p(4n + 2 - (4(p-1) + \epsilon_1)) = p(4(n-p) + 6 - \epsilon_1);$$

$$4(p-1) + 6 - \epsilon_1 \quad p(4n + 2 - (4(p-1) + 6 - \epsilon_1)) = p(4(n-p) + \epsilon_1);$$

$$4(p-1) + 2 + \epsilon_1 \quad p(4n + 2 - (4(p-1) + \epsilon_1 + 2)) = p(4(n-p) + 4 - \epsilon_1);$$

$$4(p-1) + 8 - \epsilon_1 \quad p(4n + 2 - (4(p-1) + 8 - \epsilon_1)) = p(4(n-p) - 2 + \epsilon_1).$$

The sum of these terms is $8((2n+1)p - 2p^2)$. Summing over all intervals we get

$$\sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(4n + 2 - E_{j,p}) = 8 \sum_{p=1}^{n} ((2n+1)p - 2p^2) = 8 \left( (2n+1) \frac{n(n+1)}{2} - 2 \left( \frac{(n+1)^3}{3} - \frac{(n+1)^2}{2} + \frac{(n+1)}{6} \right) \right) = 8 \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right).$$
Finally, to treat $\lambda = 4n$, we split $[2, 4n] = \bigcup_{p=1}^{n-1} [4p - 2, 4p + 2] \cup [4n - 2, 4n]$. On each of the subintervals $[4p - 2, 4p + 2]$, where the multiplicity is $p$, we note that

$$
\begin{align*}
4(p-1) + \epsilon_1 & \quad \text{gives} \quad p(4(n-p) + 4 - \epsilon_1) \\
4(p-1) + 6 - \epsilon_1 & \quad p(4(n-p) - 2 + \epsilon_1) \\
4(p-1) + 2 + \epsilon_1 & \quad p(4(n-p) + 2 - \epsilon_1) \\
4(p-1) + 8 - \epsilon_1 & \quad p(4(n-p) - 4 + \epsilon_1).
\end{align*}
$$

These terms sum to $16p(n-p)$, and in all we get

$$
\sum_{p=1}^{n-1} 16p(n-p) = \frac{8n}{3} (n^2 - 1).
$$

Finally on $[4n - 2, 4n]$ the eigenvalues $4n - 4 + \epsilon_1$ and $4n + 2 - \epsilon_1$, each with multiplicity $n$, contribute

$$
n(4n - 4 + \epsilon_1)) + n(4n - 4 + \epsilon_1)) = 2n.
$$

Thus,

$$
\sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(4n - E_{j,p})_+ = \frac{8n}{3} (n^2 - 1) + 2n = \frac{n^3}{3} - \frac{n}{12} \quad (3.4)
$$

If we substitute $n$ as a function of $\lambda$ in (3.3) or (3.4), we obtain

$$
\sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(\lambda - E_{j,p})_+ = \frac{\lambda^3}{24} - \frac{\lambda}{6} \quad \text{if } \lambda = 2, 4, 6, \ldots \quad (3.5)
$$

in both cases. (Actually (3.3) and (3.4) are only valid if $n \geq 1$ but a simple calculation shows that $\lambda = 2$ need not be excluded.) In the intervals between even integers we can prove the same thing by convexity. The Lieb-Thirring sum is a piecewise affine function of $\lambda$, and since the first-order coefficient equals the number of eigenvalues below $\lambda$, it is also convex. Assume that $\lambda$ is an even integer and let $\lambda = \hat{\lambda} + 2t$, $0 < t < 1$. By Jensen’s inequality,

$$
\begin{align*}
\sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(\lambda - E_{j,p})_+ & \leq \left( \frac{\hat{\lambda}^3}{24} - \frac{\hat{\lambda}}{6} \right)(1 - t) + \left( \frac{(\hat{\lambda} + 2)^3}{24} - \frac{\hat{\lambda} + 2}{6} \right) t \\
& = \frac{(\lambda + 2t)^3}{24} + h(t), \quad \text{where } h(t) = -\frac{t^3}{3} + \hat{\lambda} \left( -\frac{t^2}{2} + \frac{t}{2} - \frac{1}{6} \right).
\end{align*}
$$

Noting that $h'(t) = -t^2 + \lambda(1/2 - t)$ we see that $h$ has a local maximum in $(0, 1)$, namely

$$
h \left( \frac{1}{1 + \sqrt{1 + 2/\lambda}} \right) = -\frac{\lambda + 2}{6(1 + \sqrt{1 + 2/\lambda})^2} < 0 \quad \forall \lambda \geq \epsilon_1 > 2.
$$

Hence,

$$
\sum_{j=1}^{2} \sum_{p=0}^{\infty} N(p)(\lambda - E_{j,p})_+ \leq \frac{\lambda^3}{24} \left( 1 - \frac{4(\hat{\lambda} + 2)}{\lambda^3(1 + \sqrt{1 + 2/\lambda})^2} \right).
$$

The last factor will tend to one as $\lambda \to \infty$, which proves

**Theorem 3.** When $\gamma = 1$, inequality (3.1) is sharp with $R_1 = 1$. 
Part 2. Coulomb scalar potential

4. Spectrum and eigenfunctions

Treating now the case of Coulomb scalar potential, we will see that the eigenvalue problem can again be reduced to Whittaker’s equation. The spectrum is, however, very dissimilar to what was found in the case of quadratic potential.

4.1. Preparations. Using again the decomposition (2.1), we obtain the differential expression

$$H(\vec{A}, V) = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( i \frac{\partial}{\partial \theta} + \alpha \right) - \frac{\beta}{r} = \bigoplus_{m \in \mathbb{Z}} (H_m \otimes I_m),$$

(4.1)

where

$$H_m = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2}(\alpha - m)^2 - \frac{\beta}{r}.$$ 

As earlier, the first-order term can be removed by unitary equivalence under the mapping (2.3). One then obtains

$$\tilde{H}_m = U H_m U^{-1} = -\frac{d^2}{dr^2} + \frac{(\alpha - m)^2 - \frac{1}{4} - \frac{\beta}{r}}{r^2}.$$ 

This allows us to define the quadratic form of the operator in the Coulomb case:

$$\tilde{a}[u] = \sum_{m=-\infty}^{\infty} \tilde{a}_m[u_m],$$

(4.2)

where

$$\tilde{a}_m[u] = \int_0^\infty \left( \frac{du}{dr} \right)^2 + \frac{(\alpha - m)^2 - \frac{1}{4} - \frac{\beta}{r}}{r^2} |u|^2 \, dr.$$ 

(4.3)

Using the one-dimensional classical Hardy inequality,

$$\int_0^\infty |f'|^2 \, dr \leq \int_0^\infty |f|^2 \, dr \quad \forall f \in H^1_0(\mathbb{R}^+).$$

one can easily show that the quadratic form (4.2) is lower semibounded and closed on the domain $H^1_0(\mathbb{R}^2)$. This observation will simplify the examination of which formal solutions are actually eigenfunctions of the Friedrichs extension of (4.1). We then merely have to verify that the solutions belong to $H^1_0(\mathbb{R}^2)$.

4.2. Eigenfunctions. We now turn to the equation

$$\tilde{H}_m \tilde{\varphi}_m = E \tilde{\varphi}_m, \quad \tilde{\varphi}_m = U \varphi_m.$$ 

It is equivalent to

$$\tilde{\varphi}_m''(r) - 4E \left( \frac{1}{4} - \frac{\beta/4E}{r} - \frac{1}{4} - \frac{(\alpha - m)^2}{4E^2r^2} \right) \tilde{\varphi}_m(r) = 0.$$ 

Since the Coulomb potential is not confining, $E \geq 0$ corresponds to scattering states and so we can restrict our study to $E < 0$. We then have

$$\tilde{\varphi}_m''(r) + 4E \left( \frac{1}{4} + \frac{\beta/4E}{r} + \frac{1}{4} - \frac{(\alpha - m)^2}{4E^2r^2} \right) \tilde{\varphi}_m(r) = 0,$$

which can be rewritten as Whittaker’s equation,

$$\tilde{\varphi}_m''(z) + \left( \frac{1}{4} + \frac{\lambda}{z} + \frac{1}{4} - \frac{(\alpha - m)^2}{z^2} \right) \tilde{\varphi}_m(z) = 0,$$

with $z = 2\sqrt{|E|r}$, $\lambda = \beta/2\sqrt{|E|}$ and $\mu = |\alpha - m|$. Its solutions are (cf. Section 2.2) $M_{\lambda, \mu}(z)$ and $M_{\lambda, -\mu}(z)$, of which the latter is not defined if $2\mu \in \mathbb{Z} \setminus \{0\}$. With the new definition of $\mu$, the exceptional case occurs whenever $\alpha$ is a half-integer, but
as only either of the solutions obtained for each \( m \) is integrable, this will not cause any difficulties.

We know that a fundamental system of solutions is

\[
\tilde{\varphi}_m(r) = \frac{1}{\sqrt{r}} M_{\lambda, \pm \mu} \left( 2\sqrt{|E|r} \right).
\]

We use the same approach as in Section 2.2 to check that these functions lie in the domain of the Friedrichs extension, i.e., in the closure of \( C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \) with respect to (4.2) or, equivalently, the \( H_0^1 \) norm. For small \( r \)

\[
\tilde{\varphi}_m^\pm = O(r^{\pm \mu + \frac{1}{2}}) \quad \text{and} \quad \frac{d\tilde{\varphi}_m^\pm}{dr} = O(r^{-\frac{1}{2} |\alpha-m|}),
\]

and hence \( \tilde{\varphi}_m^- \) can be excluded for all \( m \). On the other hand, when \( r \) is large,

\[
\tilde{\varphi}_m^+ (r) = \left( \frac{e^{i\pi \lambda} \Gamma(2|\alpha-m| + 1)}{\Gamma(|\alpha-m| - \lambda + \frac{1}{2})} \right) \left( -2\sqrt{|E|r} \right)^{-\lambda} e^{\sqrt{|E|r}}
\]

\[
+ \frac{e^{i\pi (|\alpha-m| - \lambda + \frac{1}{2})} \Gamma(2|\alpha-m| + 1)}{\Gamma(|\alpha-m| + \lambda + \frac{1}{2})} \left( 2\sqrt{|E|r} \right)^{\lambda} e^{-\sqrt{|E|r}} (1 + O(r^{-1}))
\]

Repeating our argument from Section 2.2, finiteness of the quadratic form requires that

\[
|\alpha - m| - \beta = -n \iff E = -\left( \frac{\beta/2}{n + |\alpha - m| + \frac{1}{2}} \right)^2, \quad n \in \mathbb{N}.
\]

Clearly, the operator \( H_m \) has a sequence of negative, discrete eigenvalues starting at \(-\beta/2(|\alpha - m| + \frac{1}{2})^2\) and accumulating towards zero.

Winding up, we arrive at

**Theorem 4.** The \( L^2(\mathbb{R}^2) \) eigenfunctions of the operator (1.1) with

\[
\tilde{A}(x_1, x_2) = \alpha(-x_2, x_1)/|x|^2 \quad \text{and} \quad V(x_1, x_2) = -\beta/|x|,
\]

where \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \) and \( \beta \in \mathbb{R}_+ \), are

\[
\frac{e^{im\theta}}{\sqrt{r}} M_{\beta/2 \sqrt{|E(m,n)|}, |\alpha-m|} \left( 2\sqrt{|E(m,n)|}r \right),
\]

where \( m \in \mathbb{Z} \) and \( M_{\lambda, \mu} \) is defined in (2.7). The eigenvalues are

\[
E(m,n) = -\left( \frac{\beta/2}{n + |\alpha - m| + \frac{1}{2}} \right)^2, \quad n \in \mathbb{N}.
\]

The multiplicity of a given eigenvalue equals how many times it appears as \( m \) runs over \( \mathbb{Z} \) and \( n \) over \( \mathbb{N} \).

### 5. Eigenvalue inequalities

In this section we return to Lieb-Thirring’s inequality (3.1) and examine when it holds for the Aharonov-Bohm operator with Coulomb potential. Since the discrete spectrum is entirely situated on the negative real axis, only negative values of \( \lambda \) are interesting.
5.1. Right-hand side. The symbol of the operator is now
\[ \alpha(x, \xi) = \left( -\xi_1 - \frac{\alpha x_2}{|x|^2}, -\xi_2 + \frac{\alpha x_1}{|x|^2} \right)^2 - \frac{\beta}{|x|^2}. \]

Proceeding the same way as in Section 3.1 we obtain for all \( \lambda < 0 \) and \( 0 \leq \gamma < 1 \)
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (a(x, \xi) - \lambda)^\gamma \, dx \, d\xi = \beta^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( |\eta|^2 - \frac{1}{|\eta|^2} - \lambda \right)^\gamma \, dy \, d\eta
\]
\[
= (2\pi\beta)^2 \int_0^1 \frac{1}{\pi} \int_0^{\sqrt{\lambda + \frac{1}{r}}} \left( \lambda - \rho^2 + \frac{1}{r} \right)^\gamma \rho^3 d\rho \, d\theta
\]
\[
= \frac{\pi\beta^2}{2(\gamma + 1)} \int_0^{\infty} \left( \frac{2\pi\beta}{2(\gamma + 1)} \right)^{\gamma+1} r^\gamma \, dr
\]
\[
= (2\pi)^2 \left( \frac{\beta}{2} \right)^2 \frac{\gamma!}{2^\gamma \sin \gamma \pi} |\lambda|^{\gamma-1}.
\]

(To compute the last integral we used a contour situated on both sides of the branch cut.) The integral diverges for \( \gamma \geq 1 \), and then the Lieb-Thirring inequality is trivial.

5.2. Left-hand side, case \( \gamma = 0 \). As in the case of quadratic potential, we will write the eigenvalues \((4.4)\) in an “ordered” way, by giving new meaning to the notation in Section 2.3. We redefine
\[
\epsilon_1 = \min_{m \in \mathbb{Z}} |\alpha - m| + \frac{1}{2} = |\alpha - m_1| + \frac{1}{2}
\]
\[
\epsilon_2 = \min_{m_1, m \in \mathbb{Z}} |\alpha - m| + \frac{1}{2} = 2 - \epsilon_1 \geq \epsilon_1.
\]

The eigenvalues can then be written in the following way:
\[
E_{j,p} = -\left( \frac{\beta/2}{\epsilon_j + p} \right)^2, \quad j = 1, 2, \ p \in \mathbb{N},
\]
with multiplicity \( N(p) = \lfloor p/2 \rfloor + 1 \). The eigenvalues define subintervals
\[
I_{1,p} = [E_{1,p}, E_{2,p}] \quad \text{and} \quad I_{2,p} = [E_{2,p}, E_{1,p+1}],
\]
which clearly constitute a partition of the interval \([E_{1,0}, 0]\). If \( \alpha \) is a half-integer, \( E_{1,p} \) and \( E_{2,p} \) coincide so that \( I_{1,p} = \emptyset \). In the other limiting case, when \( \alpha \) approaches an integer, \( E_{2,p} \) will tend to \( E_{1,p+1} \), thus making \( I_{2,p} \) vanish.

The problem is to find a constant \( R_0 \) such that
\[
N_{\lambda} \leq R_0 \left( \frac{\beta}{2} \right)^2 |\lambda|^{-1} \quad \forall \lambda < 0 \quad (5.1)
\]
or, equivalently, to determine
\[
R_0 = \left( \frac{\beta}{2} \right)^2 \sup_{\lambda < 0} N_{\lambda} |\lambda|. \quad (5.2)
\]

By an argument similar to that in Section 3.2, \( R_0 \) is independent of \( \beta \). To simplify the calculations we therefore assume \( \beta = 2 \).

We first consider \( p = 0 \). On \( I_{1,0} \) we have \( N_{\lambda} = 1 \), so that necessarily
\[
R_0 \geq \sup_{I_{1,0}} N_{\lambda} |\lambda| = |E_{1,0}| = \frac{1}{\epsilon_1^2} \quad (5.3)
\]
From \( I_{2,0} \), where \( N_{\lambda} = 2 \), we obtain the lower bound
\[
R_0 \geq \sup_{I_{2,0}} N_{\lambda} |\lambda| = 2|E_{2,0}| = \frac{2}{(2 - \epsilon_1)^2} \quad (5.4)
\]
Actually the supremum will always be attained either on $I_{1,0}$ or $I_{2,0}$. Too see this we will prove an upper bound on such values of $R_0$ that are obtained upon maximising (5.2) with $\lambda$ restricted to intervals $I_{j,p}$, $p \geq 1$. We have

$$N_\lambda = \begin{cases} \sum_{q=0}^{p-1} 2(\lfloor q/2 \rfloor + 1) + \lfloor p/2 \rfloor + 1 & \text{if } \lambda \in I_{1,p}, \\ \sum_{q=0}^{p} 2(\lfloor q/2 \rfloor + 1) & \text{if } \lambda \in I_{2,p}. \end{cases}$$

Since $\sum_{q=0}^p \lfloor q/2 \rfloor \leq p^2/4$, we readily obtain

$$N_\lambda|\lambda| \leq \frac{p^2 + 3p + 3}{2} \left( \frac{1}{\epsilon_1 + p} \right)^2 \text{ if } \lambda \in I_{1,p} \quad (5.5)$$

and

$$N_\lambda|\lambda| \leq \frac{(p+2)^2}{2} \left( \frac{1}{\epsilon_2 + p} \right)^2 = \frac{1}{2} \left( \frac{1}{1 - \frac{\epsilon_1}{p^2}} \right)^2 \text{ if } \lambda \in I_{2,p}. \quad (5.6)$$

Clearly both these bounds decrease as functions of $p$. It is also easy to verify that the value of $R_0$ as in (5.3) and (5.4) is always greater than that in (5.5) and (5.6) (with $p = 1$) for a given $\epsilon_1$. Hence,

$$R_0 = \max \left\{ \frac{1}{\epsilon_1^2 + (2 - \epsilon_1)^2} \right\}.$$ 

Writing $\epsilon_1$ explicitly, we can state

**Theorem 5.** When $\gamma = 0$, inequality (3.1) is sharp with

$$R_0 = \begin{cases} \frac{1}{(\frac{1}{2} + |\alpha - m_1|)^2} & \text{if } 0 < |\alpha - m_1| \leq 2\sqrt{2} - \frac{5}{2}, \\ \frac{2}{(\frac{1}{2} - |\alpha - m_1|)^2} & \text{if } 2\sqrt{2} - \frac{5}{2} \leq |\alpha - m_1| \leq \frac{1}{2}. \end{cases}$$

We note that $R_0 \geq (\sqrt{2} + 1)/2 \approx 1.207$ and $R_0 \uparrow$ when $\alpha$ tends to an integer. Another remark is that the leading term in the expansion of $N_\alpha$ is $\frac{1}{2} \lambda$, independently of $\alpha$ and $\beta$. This fact is suggested by the bounds (5.5) and (5.6) and we have been able to verify it by deriving closed expressions for finite sums over the multiplicities. Due to the positive higher-order terms, $R_0$ is however strongly influenced by the location of the lowest eigenvalues.

**References**


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