Representing matroids by polynomials with the half-plane property

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2000 *Mathematics Subject Classification.* 68R05,05B35

Key words and phrases. Matroid, Matroid representation, Valuated matroid, Tutte-group, Stable polynomial, Half-plane property, Rayleigh property

Abstract. A matroid $\mathcal{M}$ is said to have the weak half-plane property (*wHPP*) if there exists a stable multiaffine homogeneous complex polynomial f with support equal to the set of bases of $\mathcal{M}$. This is a generalization of the half-plane property (*HPP*), where we require that all the coefficients of f are equal to zero or one. Both properties were recently treated by Choe, Oxley, Sokal and Wagner in [**COSW04**]. In [**Brä07**], Brändén proved that not every matroid is *wHPP* by showing that the Fano matroid F_7 is not. We provide two new proofs of the fact that F_7 is not a *wHPP*-matroid. We investigate and state conditions for when *wHPP*=*HPP* for $\mathcal{M}$. We use concepts and techniques developed for the Tutte-group of a matroid and valuated matroids by Dress, Wenzel and Murota to prove that the projective geometry matroids $PG(r-1,q)$ are not *wHPP* and that a binary matroid is a *wHPP*-matroid if and only if it is regular. This shows that there exist large families of matroids that are not *wHPP*. We answer questions posed by Choe et al., by proving that the coextensions $AG(3,2)$ and S_8 of F_7, and the matroids T_8 and R_9, are not *wHPP*, extending the answer given by [**Brä07**].

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CHAPTER 1

Introduction, concepts and definitions

A matroid is a discrete mathematical object that captures the essence of the notion of independence from linear algebra and also abstracts the notion of cycles from graph theory. It is a problem of main interest in matroid theory to find different ways of representing matroids as to provide a clear and compact form describing their structure, and to equip the theory with results borrowed from the fields where these representations exist. A new way of representing matroids was recently treated in depth in [**COSW04**], and relates the set of bases of a matroid with the support of homogeneous multiaffine polynomials with prescribed non-vanishing properties (stable polynomials). It is known that such polynomials always represent a matroid but since very recently it is known that not all matroids can be represented in this way. Hence, it is a challenging problem to determine if a given matroid is representable in this way or not.

Our goal will be to describe and refine recent techniques, treated in [**COSW04, Brä07**], to determine if a matroid is representable or not in the above sense and to use them to study the representability of specific families of matroids.

1.1. Matroids and their representations

A *matroid* is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E such that:

(1) $\mathcal{I}$ is *non-empty*,
(2) $\mathcal{I}$ is *hereditary*, i.e., if $U \in \mathcal{I}$ and $V \subseteq U$, then $V \in \mathcal{I}$,
(3) $\mathcal{I}$ satisfies the *independence augmentation axiom*, i.e., if $U, V \in \mathcal{I}$ and $|U| < |V|$ then $\exists x \in V \setminus U$ such that $U \cup \{x\} \in \mathcal{I}$.

The set E is called the *ground set* of the matroid and $\mathcal{I}$ is the class of *independent sets*. The subsets of E that are not in $\mathcal{I}$ (i.e. $2^E \setminus \mathcal{I}$) are called *dependent sets*. We will sometimes say $\mathcal{M}$ instead of $(E, \mathcal{I})$ in order to make reference to an arbitrary matroid.

The motivating example of a matroid is the following. Let $E = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space over some field $\mathbb{F}$ and $\mathcal{I}$ the collection of subsets of E that are linearly independent in this field. Clearly, $(E, \mathcal{I})$ satisfies properties 1, 2 and 3. We can go further and notice that we can construct the matrix $A = [v_1 v_2 \ldots v_n]$ where v_i are the columns of A and that $(E, \mathcal{I})$ now can be viewed as the matroid generated by the independent columns of a matrix. And we can even "forget" that E is a set of vectors and consider it just as the set of indices of these vectors, so let $E' = [n]$ and let $\mathcal{I}'$ be the class of subsets of $[n]$ that are indices for some independent set in $\mathcal{I}'$. This construction represents a bijection from E to E' that preserves the independent sets and then we call $(E, \mathcal{I})$

and $(E', \mathcal{I}')$ to be *isomorphic* and regard them as being the same matroid. The matroid that we just discussed is called the *vector matroid* of A and is also denoted as $M[A]$. We say also that A represents $(E, \mathcal{I})$ since from its columns and the relations that hold between them as elements of a vector space over the field $\mathbb{F}$ we have enough information to reconstruct the set E and the class $\mathcal{I}$. There are matroids that cannot be represented by matrices. We will say that a matroid $\mathcal{M}$ is $\mathbb{F}$-*representable* or simply *representable* if there exists $A \in M_{m\times n}(\mathbb{F})$ for some $m, n \in \mathbb{N}$ and some field $\mathbb{F}$ such that $\mathcal{M} = M[A]$. Consider the following example:

EXAMPLE 1.1. The matroid $\mathcal{M} = ([3], \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\})$ can be represented by the columns of the matrix $A \in M_{3\times 3}(\mathbb{R})$:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

With this matrix we see that $\mathcal{M} = M[A]$ so $\mathcal{M}$ is $\mathbb{R}$-representable. The matrix $B \in M_{3\times 3}(GF(2))$:

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

also satisfies $\mathcal{M} = M[B]$. So $\mathcal{M}$ is also $GF(2)$-representable.

The representable matroids constitute an interesting class of matroids and will play an important role in our study of the representation of matroids with stable polynomials since, as we will show in Chapter 2, the matroids that are $\mathbb{C}$-representable, in particular, also have a representation with stable polynomials. The matroids that are $GF(2)$-representable are called *binary* and the $GF(3)$-representable are called *ternary*. A matroid that is representable by a *totally unimodular matrix* over $\mathbb{R}$, i.e. a matrix for which all its subdeterminants are in the set $\{-1, 0, 1\}$, is called *regular* or *unimodular matroid*. A matroid that can be represented over the complex numbers by a matrix for which all its subdeterminants are complex sixth-roots of unity or zero, is called a *sixth-root-of-unity matroid* or $\sqrt[6]{1}$-*matroid*. Some of these classes of matroids are related by the following strong theorems, the proof of it can be found in [**Oxl06**] and [**Whi97**]:

Theorem 1.2 ([**Oxl06**] Theorem 6.6.3)**.** *The Following statements are equivalent for a matroid $\mathcal{M}$:*

(i) *$\mathcal{M}$ is regular.*
(ii) *$\mathcal{M}$ is representable over every field.*
(iii) *$\mathcal{M}$ is binary and, for some field $\mathbb{F}$ of characteristic other than two, $\mathcal{M}$ is $\mathbb{F}$-representable.*

Theorem 1.3 ([**Whi97**] Theorem 1.2)**.** *The Following statements are equivalent for a matroid $\mathcal{M}$:*

(i) *$\mathcal{M}$ is a $\sqrt[6]{1}$-matroid.*
(ii) *$\mathcal{M}$ is representable over $GF(3)$ and $GF(4)$.*
(iii) *$\mathcal{M}$ is representable over $GF(3)$ and $GF(2^k)$ for some positive even integer k.*

Since a matroid is an abstraction of the concept of independence, it is valid to ask ourselves what are the maximal independent sets with respect to inclusion. As a consequence of Condition 3 in the definition, it is easy to prove that all the maximal independent sets have the same cardinality. The common cardinality of the maximal independent sets is called the *rank* of the matroid and for any $A \subseteq E$, we define $\text{rank}(A) = \max\{|I| : I \subseteq A,\ I \in \mathcal{I}\}$, which coincides with the rank of the matroid if A contains a maximal independent set. We call a maximal independent set a *basis* or *base* for the matroid and because the hereditary condition in the definition of a matroid it follows that the class of bases, that we will write $\mathcal{B}$, is enough to represent a matroid. In fact, an equivalent definition for a matroid is:

Definition 1.4. Let $\mathcal{B}$ be a class of subsets of a finite set E such that:

(1) $\mathcal{B}$ is *non-empty*.
(2) $\mathcal{B}$ satisfies *the basis exchange axiom*, i.e., $\forall U, V \in \mathcal{B}$ and $x \in U \setminus V$ $\exists y \in V \setminus U$ such that $U \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

Then a *matroid* is a pair $(E, \mathcal{I})$ where $I \in \mathcal{I}$ if I is a subset of any element in $\mathcal{B}$ and $\mathcal{B}$ is called the set of *bases* of the matroid.

A *circuit* is a minimal dependent set in the sense that $C \subseteq E$ and $C \notin \mathcal{I}$, but $C \setminus \{i\} \in \mathcal{I}$ for all $i \in C$. A circuit with only one element is called a *loop* and the members of a circuit with two elements are called *parallel*. A matroid that has no loops or parallel elements is called *simple*. If B is a basis and $i \in E \setminus B$, then one can prove that there is a unique circuit $C(i, B)$ that is contained in $B \cup \{i\}$. This circuit is called *the fundamental circuit of i with respect to B*. It is also the case that the class of circuits determines the matroid. There are many equivalent definitions of a matroid. It can be defined in terms of its independent sets, bases, circuits, rank function, closure operator, etc. The reader interested in these definitions, and the proofs of equivalence, can find a comprehensive treaty in [**Oxl06**].

As we mentioned before, the concept of a matroid also generalizes a dependence relation that is found between the edges of a graph. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $\mathcal{M} = (E, \mathcal{I})$ be the pair with $\mathcal{I}$ defined by all the subsets of E that contain no cycles, i.e., the forests. It is easy to verify that $\mathcal{M}$ satisfy the three conditions in our initial definition of a matroid. The bases for this matroid are the spanning forests (spanning trees if the graph is connected). We can then say that the graph G is a representation of the matroid $\mathcal{M}$ and we write $\mathcal{M} = M[G]$. The matroids that arise in this way are called *graphic*. We can represent a graph using its *incidence matrix*. This matrix is a representation of $M[G]$ over $GF(2)$, so the graphic matroids are also binary. In fact, they are also ternary, which by Theorem 1.2 implies that graphic matroids are representable over any field. Example 1.1 is an example of a graphic matroid and Figure 1.1 shows a graph that gives rise to this matroid.

Matroids have many compact and practical representations. We will use one that is very useful when we are dealing with specific matroids of small ground set and small rank. The so called *geometric representation*. This pictoric representation (that is different from the one with graphs we just discussed) arises from the fact that the set of *affinely independent* subsets of a finite set of vectors in a vector space over some field $\mathbb{F}$ constitutes a matroidal structure known as an *affine matroid*. For our purpose we will just describe how a matroid, of rank at most 4, can be

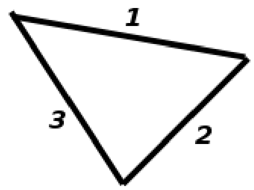


Figure 1.1: A graph that represents the same matroid as in Example 1.1.

represented and define the conditions for one of this geometric diagrams to be the representation of a matroid.

Geometric Representation. For a matroid $\mathcal{M} = (E, \mathcal{I})$ of rank at most 4 we follow the rules:

(1) All loops are represented as points with the element as a label and located in a box apart from the rest of the diagram.
(2) Parallel elements are represented by a single point with all the labels.
(3) Any other element that is not a loop and has not a parallel peer is represented by a particular point labelled with the element.
(4) Any circuit of rank 2 is represented as all points being collinear (lines need not to be straight).
(5) Any circuit of rank 3 is represented as all points being coplanar.
(6) Any circuit of rank 4 is represented as all points being in the space.

The geometric representation for the matroid we discussed in Example 1.1 is depicted in Figure 1.2 and two more examples of geometric representations are illustrated for the Fano matroid (Figure 1.3a) and the Pappus matroid (Figure 1.3b).

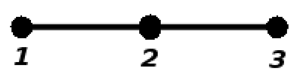
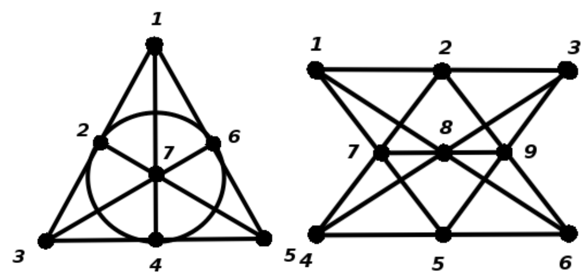


Figure 1.2: Geometric representation of the matroid in Example 1.1, also known as $U_{2,3}$.

It is important to note that not all the geometries with points, lines and planes represent a matroid. In order to do so it must satisfy the conditions:

(1) Any two distinct planes meeting in more than one point do so in a line.
(2) Any two distinct lines meeting in a point do so in at most one point and lie on a common plane.
(3) Any line not lying on a plane intersects it in at most one point.

A *hyperplane* in a matroid $\mathcal{M}$ is a subset $X \subseteq E$ such that $\text{rank}(X) = \text{rank}(\mathcal{M}) - 1$ and such that for any other element $e \in E \setminus X$ it happens that $\text{rank}(X \cup \{e\}) = \text{rank}(\mathcal{M})$. We introduce now a way of creating a matroid starting from another.



(a) F_7 (b) *Pappus*

Figure 1.3: Geometric representation of the Fano F_7 and Pappus matroids.

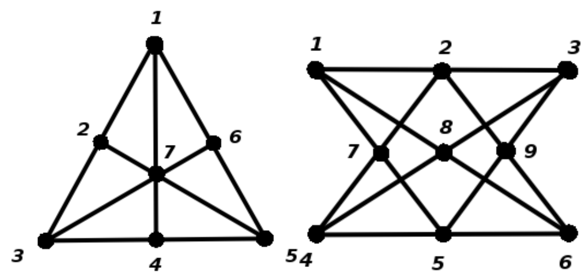
Proposition 1.5. *Let $\mathcal{M}$ be a matroid having a subset X that is both a circuit and a hyperplane. Let $\mathcal{B}' = \mathcal{B}_{\mathcal{M}} \cup X$. Then $\mathcal{B}'$ is the set of bases of a matroid $\mathcal{M}'$ on $E(\mathcal{M})$.*

PROOF. Since $\mathcal{B}$ satisfies the *basis exchange axiom*, we only have to prove that between any $B \in \mathcal{B}$ and X the axiom still holds. Let $r = \text{rank}(\mathcal{M})$ and note that, because X is a circuit, after we remove any element from it, the resulting set is independent. And since it is also a hyperplane ($\text{rank}(X) = r - 1$) this implies that $|X| = r$.
So for all $x \in X \setminus B$ the set $X \setminus \{x\}$ is independent. It follows that $r - 1 = |X \setminus \{x\}| < |B| = r$ implies by the *independence augmentation axiom* that there exists $y \in B \setminus (X \setminus \{x\})$ such that $X \setminus \{x\} \cup \{y\} \in \mathcal{I}$ and hence it also belongs to $\mathcal{B}$. Now suppose that the basis exchange axiom does not hold in the other direction. Let $x \in B \setminus X$ such that for all $y \in X \setminus B$, $\text{rank}(B \setminus \{x\} \cup \{y\}) = r - 1$. This implies that $\text{rank}(B \setminus \{x\} \cup X) = r - 1$. A contradiction with the fact that X is a hyperplane. Then the axiom must hold for all $\mathcal{B}'$ implying that is the set of bases of a matroid. $\square$

We call the matroid resulting from Proposition 1.5 a *relaxation* of the matroid $\mathcal{M}$. In the geometric representation this can be seen as the elimination of a restriction, i.e., a line or a plane. For example, if we remove the line $\{2, 4, 6\}$ in the Fano matroid in Figure 1.3 we get the so called non-Fano matroid and if we remove the line $\{7, 8, 9\}$ in the Pappus matroid we get the non-Pappus matroid as shown in Figure 1.4

The *dual* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is the matroid $\mathcal{M}^*$ such that its set of bases is $\mathcal{B}_{\mathcal{M}^*} = \{E \setminus B : B \in \mathcal{B}_{\mathcal{M}}\}$. Clearly, $(\mathcal{M}^*)^* = \mathcal{M}$.

As in graph theory, we define the *deletion* of $i \in E$ from $\mathcal{M}$ to be the matroid $(E \setminus \{i\}, \{I \subseteq E \setminus \{i\} : I \in \mathcal{I}\})$ and we denoted it $\mathcal{M} \setminus i$. We define the *contraction* of $i \in E$ from $\mathcal{M}$ to be the matroid $(E \setminus \{i\}, \{I \subseteq E \setminus \{i\} : I \cup \{i\} \in \mathcal{I}\})$ and we denoted it $\mathcal{M}/i$. Finally, we say that a *minor* of $\mathcal{M}$ is the matroid obtained from $\mathcal{M}$ by a sequence of deletions and contractions.



(a) F_7^- (b) $non - Pappus$

Figure 1.4: Geometric representation of the non-Fano F_7^- and non-Pappus matroids.

We say that a matroid is *connected* or *2-connected* if for any pair of distinct elements in $E_{\mathcal{M}}$ there is a circuit containing both of them. Otherwise it is *disconnected*. A *connected component* of $\mathcal{M}$ is a nonempty subset $S \subseteq E_{\mathcal{M}}$ that is maximal with respect to the property that for any pair of distinct elements in S there is a circuit containing both of them. If a matroid $\mathcal{M}$ has z connected components we can write it as a *direct sum* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_z$ on disjoint sets $E_{\mathcal{M}} = E_{\mathcal{M}_1} \uplus E_{\mathcal{M}_2} \uplus \ldots \uplus E_{\mathcal{M}_z}$ and $\mathcal{B}_{\mathcal{M}} = \{B_1 \cup B_2 \cup \ldots \cup B_z : B_i \in \mathcal{B}_{\mathcal{M}_i}\}$.

Of course, we have not referred to all the properties of matroids here, even not to all the basic. Further properties of matroids are discussed in [**Oxl03, Oxl06**].

1.2. Polynomials with the half-plane property

Let $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ be a polynomial in finitely many complex variables. We will write $f(z) = \sum_{S \in \mathcal{C}} a_S z^S$ where $\mathcal{C}$ is any finite class of multisets of $E = [n]$, which we will call in analogy to our matroid definition as the *ground set*. Here $z = \{z_i\}_{i \in E}$ and $z^S = \prod_{i \in S} z_i$. The *degree* of a monomial $a_S z^S$ is the cardinality of S as a multiset and the degree of f is the largest cardinality as a multiset of $S \in \mathcal{C}$ such that a_S is nonzero. We say that f is *homogeneous* or *r-regular* if all the monomials have the same degree r. We say that f is *affine* in the variable z_i if it appears at most one time in each monomial. And f is *multiaffine* if it is affine in z_i for all $i \in E$. The *support* of f is defined as $\mathrm{supp}(f) = \{S \in \mathcal{C} : a_S \neq 0\}$.

Let $H \subset \mathbb{C}$ be any open half-plane whose boundary contains the origin. We say that $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ is H-stable, or that it has the *half-plane property* with respect to it, if $f(z) \neq 0$ whenever $z_i \in H$ for all $i \in E$. In particular we say that f is *stable* if $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and that it is *Hurwitz stable* or that it has the *right half-plane property* if $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Now let $\mathcal{C}$ be any finite class of sets or multisets of a given ground set $E = [n]$. We define

$$f_{\mathcal{C}}(z) = \sum_{S \in \mathcal{C}} z^S$$

to be the *generating polynomial* in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ of the class $\mathcal{C}$. If $\mathcal{C} = \mathcal{B}_\mathcal{M}$, the set of basis of a matroid on E, then we call $f_{\mathcal{B}_\mathcal{M}}$ the *basis generating polynomial* of the matroid $\mathcal{M}$. Notice that $f_{\mathcal{B}_\mathcal{M}}$ is homogeneous and multiaffine by the properties mentioned before for the bases of a matroid.

In [**COSW04**], they proved that if f is a homogeneous multiaffine polynomial that is stable then $\operatorname{supp}(f)$ is the set of bases for a matroid (Theorem 2.7). They also showed that there are matroids for which $f_{\mathcal{B}_\mathcal{M}}$ is not stable. It is worthy to note that if f is homogeneous and multiaffine and is H_o-stable for a particular half-plane H_o then it is also H-stable for any half-plane H. Indeed, we can make a simultaneous rotation of all the variables (a change of variable $z_i' = z_i e^{i\theta}$ for all $i \in E$). So, in the following, we may just refer to stability when we discuss r-regular multiaffine homogeneous polynomials in order to make reference of H-stability for any half-plane H. Results that we will discuss in more depth in Chapter 2 motivate the following definition:

Definition 1.6 (Half-plane property of a matroid). Let $\mathcal{M}$ be a matroid. If $f_{\mathcal{B}_\mathcal{M}}$ is stable then we say that $\mathcal{M}$ has the *half-plane property* or that it is *HPP-representable* or that it is a *HPP-matroid*.

Since there are matroids for which the HPP does not hold, it is worthy to ask: Does there for any matroid $\mathcal{M}$ exist a homogeneous multiaffine polynomial f such that $\operatorname{supp}(f) = \mathcal{B}_\mathcal{M}$ and f stable?. That is, if we relax the condition that the coefficients of f must be in the set $\{0, 1\}$ to a bigger set like $\mathbb{C}$, is there a stable polynomial that represents $\mathcal{M}$?. The answer given by [**Brä07**] is negative for this question also. There he showed that this is not true for the Fano matroid. Hence it is also interesting to identify which matroids have the weakest version of this representation. We define:

Definition 1.7 (Weak half-plane property of a matroid). Let $\mathcal{M}$ be a matroid. If there exist a stable polynomial $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ such that $\operatorname{supp}(f) = \mathcal{B}_\mathcal{M}$ then we say that $\mathcal{M}$ has the *weak half-plane property* or that it is *wHPP-representable* or that it is a *wHPP-matroid*.

It is important to remark here that in the proof of [**Brä07**], he showed that for the Fano matroid *wHPP=HPP*. It is the main goal of this work to study some other cases where the reduction of *wHPP* to *HPP* can be done, and this will be discussed in Chapter 3.

1.3. Valuated matroids

Valuated matroids were introduced by Dress and Wenzel in [**DW90b, DW92**] to study a variant of the *greedy algorithm* for weight functions defined on the class $\binom{E}{r}$ of a ground set E.

Let $\mathcal{G}$ be a totally ordered abelian group and consider an element, say $\{-\infty\}$, such that $-\infty \leq x$ and $-\infty + x = -\infty$ for all $x \in \mathcal{G} \cup \{-\infty\}$. A *valuated matroid* $\mathcal{M}$ of rank r is defined to be a pair $\mathcal{M} = (E, v)$ that satisfies:

(1) $v : \binom{E}{r} \to \mathcal{G} \cup \{-\infty\}$ is a function such that $v(B) \neq -\infty$ for at least some $B \in \binom{E}{r}$.

(2) $\forall B_1, B_2 \in \binom{E}{r}$ and $\forall i \in B_1 \setminus B_2, \exists j \in B_2 \setminus B_1$ such that

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{i\}).$$

In particular in [**DW90b**] they considered $\mathcal{G} = \mathbb{R}$ as we also will consider in this work. This definition is a generalization of a matroid, since the set $\mathcal{B} = \{B \in \binom{E}{r} : v(B) \neq -\infty\}$, is the set of basis of a matroid. Condition 2 is a variant of the *strong matroid basis exchange axiom*. And a function

$$v : \mathcal{B}_{\mathcal{M}} \to \mathbb{R}, v(B) = -\infty \text{ if } B \notin \mathcal{B}$$

is called a *valuation* of $\mathcal{M}$ if v satisfies: $\forall B_1, B_2 \in \mathcal{B}$ and $\forall i \in B_1 \setminus B_2, \exists j \in B_2 \setminus B_1$ such that

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{i\}) \tag{1.1}$$

We will refer to Condition 1.1 as the *exchange condition for valuations* and has according to [**Mur96**] some equivalent formulations:

Theorem 1.8 ([**Mur96**] Theorem 1.1)**.** *Let $\mathcal{M}$ be a matroid. For $v : \mathcal{B}_{\mathcal{M}} \to \mathbb{R}$ The following conditions are equivalent:*

(i) $\forall B_1, B_2 \in \mathcal{B},\ \forall i \in B_1 \setminus B_2,\ \exists j \in B_2 \setminus B_1$ *such that*

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{i\}).$$

(ii) $\forall B_1, B_2 \in \mathcal{B},\ B_1 \neq B_2,\ \exists i \in B_1 \setminus B_2,\ \exists j \in B_2 \setminus B_1$ *such that*

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{i\}).$$

(iii) $\forall B_1, B_2 \in \mathcal{B},\ \forall i \in B_1 \setminus B_2,\ \exists j \in B_2 \setminus B_1,\ \exists k \in B_1 \setminus B_2$ *such that*

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{k\}).$$

Two valuations $v_1, v_2 : \binom{E}{r} \to \mathcal{G} \cup \{-\infty\}$ are called *projectively equivalent* if and only if there exist a $\alpha \in \mathcal{G}$ and a function $g : E \to \mathcal{G}$ such that for all $B \in \binom{E}{r}$ it holds that

$$v_2(B) = \alpha + \sum_{i \in B} g(i) + v_1(B).$$

If $\mathcal{G} = \mathbb{F}$, a field, then it is equivalent to the relation

$$v_2(B) = \sum_{i \in B} g(i) + v_1(B).$$

Every matroid has at least a *trivial* valuation that assigns 0 to every basis element and $-\infty$ elsewhere. A matroid $\mathcal{M}$ is called *rigid* if all the valuations of $\mathcal{M}$ are projectively equivalent to the trivial valuation.

The exchange condition for valuations can be replaced with a local condition that is so much easier to verify. Theorem 1.9 can be derived from the Theorem 2.3 in [**Mur06**] by considering the function $v = -f$ and the particular case of the set of bases of a matroid instead of the more generic case proved there for jump systems.

Theorem 1.9 (corollary from [**Mur06**] Theorem 2.3)**.** *The exchange condition for valuations 1.1 is equivalent to:*

$\forall B_1, B_2 \in \mathcal{B}$ *with* $|B_1 \Delta B_2| = 4$, $\exists \{i, j\} \in B_1 \Delta B_2$ *such that* $B_1 \setminus \{i\} \cup \{j\} \in \mathcal{B}$ *and* $B_2 \setminus \{j\} \cup \{i\} \in \mathcal{B}$ *and*

$$v(B_1) + v(B_2) \leq v(B_1 \setminus \{i\} \cup \{j\}) + v(B_2 \setminus \{j\} \cup \{i\}). \tag{1.2}$$

We will refer from now on to Condition 1.2 as the *local exchange condition for valuations*.

CHAPTER 2

Representability of matroids by stable polynomials

As we mentioned in Chapter 1, a *wHPP-matroid* can be represented by a homogeneous multiaffine stable complex polynomial. From the point of view of matroid theory is interesting to determine the transformations that leave this property invariant. We will here discuss a collection of important theorems and results concerning the half-plane and the weak half-plane properties. Some of the results that are found in the literature are more general than what we need for further discussion. We will just attain our vision to these particular properties. However, the theorems and results are sometimes stated as general as originally proved.

2.1. Operations that preserve the stability of polynomials

We are considering a polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ that is H-stable for some half-plane H.

Proposition 2.1. *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be H-stable then:*

(i) *The dual polynomial $f^*(z) = z^E f(1/z)$ is $\overline{H}$-stable, where $\overline{H} = \{\overline{z} : z \in H\}$ and $1/z = \{1/z_i\}_{i \in E}$.*
(ii) *If f is multiaffine and homogeneous then f^* is stable.*

Proposition 2.2 ([**COSW04**] Proposition 4.2)**.** *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial such that* $\operatorname{supp}(f) = \mathcal{B}_{\mathcal{M}}$*, then f^* has support* $\operatorname{supp}(f^*) = \mathcal{B}_{\mathcal{M}^*}$*, where $\mathcal{M}^*$ is the matroid dual to $\mathcal{M}$.*

The last two propositions imply that the stability properties for a matroid (*HPP* and *wHPP*), are preserved by taking duals.

Without losing stability we can fix some variables of f with values in $\operatorname{cl}(H)$ (the closure of H).

Proposition 2.3 ([**COSW04**] Proposition 2.1)**.** *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be H-stable and let $f_{z_i=k}$ be the polynomial obtained from f by fixing the value of the variable z_i to $k \in \operatorname{cl}(H)$. Then $f_{z_i=k}$ is H-stable or identically zero.*

If, in particular, we set $z_i = 0$ for f homogeneous and multiaffine, then $f_{z_i=0}$ is stable or identically zero and

Proposition 2.4 ([**COSW04**] Proposition 4.1(d))**.** *Let $\mathcal{M} = (E, \mathcal{I})$ and $f \in \mathbb{C}[z_1, \dots, z_n]$ be such that* $\operatorname{supp}(f) = \mathcal{B}_{\mathcal{M}}$*. Then* $\operatorname{supp}(f_{z_i=0}) = \mathcal{B}(\mathcal{M} \setminus i)$.

This means that for any series of deletions of elements from a *wHPP-matroid* $\mathcal{M}$, we will be able to find a stable polynomial that represents the resulting matroid. Furthermore, if we already have a stable polynomial f that represents $\mathcal{M}$, we can obtain the new polynomial by setting the deleted variables to zero. We would suspect that something similar should be the case for the matroids that are obtained by a series of contractions from a *wHPP-matroid*, and it is. Another useful operation that preserves stability is that of taking partial derivatives as (see also [**Brä07**]).

Proposition 2.5 ([**COSW04**] Proposition 2.8 (a) and Remarks)**.** *Let* $f \in \mathbb{C}[z_1, \dots, z_n]$ *be Hurwitz stable. Then:*

(i) *Either* $\partial f/\partial z_1 = 0$ *or* $\partial f/\partial z_1$ *is Hurwitz stable.*
(ii) *The polynomial* $\alpha f + \sum_{i \in E} \lambda_i \partial f/\partial z_i$ *for* $\alpha, \lambda_i \in \mathbb{R}^+$*is Hurwitz stable.*
(iii) *If* $\alpha = (\alpha_1, \dots, \alpha_n)$*, then*

$$\partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} f$$

is identically zero or is Hurwitz stable.

A stable polynomial that represents a contraction of a matroid $\mathcal{M}$ can be a derivative of a stable polynomial that represents $\mathcal{M}$.

Proposition 2.6 ([**COSW04**] Proposition 4.1 (c))**.** *Let* $\mathcal{M} = (E, \mathcal{I})$ *and* $f \in \mathbb{C}[z_1, \dots, z_n]$ *be such that* $\operatorname{supp}(f) = \mathcal{B}_{\mathcal{M}}$*. Then* $\operatorname{supp}(\dfrac{\partial f}{\partial z_i}) = \mathcal{B}(\mathcal{M}/i)$.

The last two operations will allow one to reduce the problem of finding out if a matroid has the *HPP* or the *wHPP* to the one of one of its minors.

2.2. Necessary conditions for polynomials to be H-stable

The first necessary condition for a polynomial to be H-stable is precisely the one that creates the strong relation between stable polynomials and matroids.

Theorem 2.7 ([**COSW04**] Theorem 7.1 ,[**Brä07**] Corollary 3.4)**.** *Let* $f \in \mathbb{C}[z_1, \dots, z_n]$ *be a homogeneous multiaffine stable polynomial. Then* $\operatorname{supp}(f)$ *is the set of bases of a matroid.*

In fact in [**Brä07**], he proved the more general result that for any polynomial f that is H-stable $\operatorname{supp}(f)$ is a *jump system*. This has as a corollary Theorem 2.7.

One striking necessary condition proved in [**COSW04**] is that all the complex coefficients of a homogeneous polynomial with the half-plane property have the same phase. This result simplifies the study of the weak half-plane property since we can disregard the coefficients as complex numbers and assume that all of the coefficients in the support of a polynomial with the property are real and positive.

Theorem 2.8 ([**COSW04**] Theorem 6.1)**.** *Let* $f(z) = \sum_{S \in M} a_S z^S$ *be a homogeneous polynomial of degree* r *in* $\mathbb{C}[z_1, \dots, z_n]$*. If* f *has the half-plane property, then all the non-zero coefficients* a_S*, i.e.* $S \in \operatorname{supp}(f)$*, have the same phase.*

Let us now define the *Rayleigh difference* $\Delta_{ij}(f)$ for $f \in \mathbb{C}[z_1, \dots, z_n]$ and $i, j \in [n]$

$$\Delta_{ij}(f) = \frac{\partial f}{\partial z_i} . \frac{\partial f}{\partial z_j} - \frac{\partial^2 f}{\partial z_i \partial z_j} . f = -f^2 \frac{\partial^2}{\partial z_i \partial z_j} [\log |f|].$$

Theorem 2.9 ([**Brä07**] Theorem 5.10)**.** *Let* $f \in \mathbb{R}[z_1, \dots, z_n]$ *be multiaffine. Then the following are equivalent:*

(i) *For all* $x \in \mathbb{R}^n$ *and* $i, j \in [n]$*,* $\Delta_{ij}(f)(x) \geq 0$.

(ii) *f is stable.*

A polynomial that satisfies the first statement in Theorem 2.9, is also called *strongly Rayleigh*. Hence, the necessary and sufficient condition in Theorem 2.9 can be rewritten as: f is stable if and only if it is strongly Rayleigh. In [**WW09**] they showed that the two statements in Theorem 2.9 are also equivalent to:

(iii) For every index $i \in E$, both $f_{z_i=0}$ and $\partial f/\partial z_i$ are strongly Rayleigh, and for some pair of indices $\{i,j\} \subseteq E$, $\Delta_{ij}(f)(x) \geq 0$ for $x \in \mathbb{R}^n$.

The latter condition is useful when proving stability through an induction on $|E|$ since it reduces to the verification of only one Rayleigh difference.

There is one more necessary condition derived from Theorem 2.9 by [**Brä07**]. It is stated and proved in Chapter 3 as Theorem 3.1. This new condition gives a set of restrictions for the coefficients of a multiaffine homogeneous stable polynomial.

There is also a way to reduce the verification of the stability of f to the case of a univariate complex polynomial. The following necessary and sufficient condition is helpful when finding counterexamples to the stability of a homogeneous polynomial f.

Proposition 2.10 ([**COSW04**] Proposition 5.2)**.** *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous. For $x, y \in \mathbb{R}^n$ define the univariate complex polynomial $f_{x,y}(\xi) = f(\xi x + y)$. Then the following are equivalent:*

(i) *f is stable.*
(ii) *$\forall x, y \geq 0$ with $x + y > 0$, all the roots of $f_{x,y}$ lie in $(-\infty, 0]$.*
(iii) *$\forall x, y > 0$, all the roots of $f_{x,y}$ lie in $(-\infty, 0]$.*

2.3. Sufficient conditions for stability

The determinant condition stated and proved in [**COSW04**] categorizes the family of $\mathbb{C}$-representable matroids inside the *wHPP*-matroids and also defines a way to find a stable polynomial that represents such matroids.

Theorem 2.11 (Determinant condition [**COSW04**] Theorem 8.1)**.** *Let A be an arbitrary complex $r \times n$ matrix, and define Q_A by*

$$Q_A(z) = \det(AZA^*), \textit{where } Z = \operatorname{diag}(z_1, \ldots, z_n) \textit{ and } {}^*\textit{denotes Hermitian conjugate}$$

Then:

(i) *Q_A is stable.*
(ii) *$Q_A = \sum_{\substack{S \subseteq [n] \\ |S|=r}} |\det(A|S)|^2 z^S$, where $A|S$ denotes the square submatrix of A using the columns indexed by the set S.*

Corollary 2.12 ([**COSW04**] Corollary 8.2)**.**

(i) *Every matroid representable over $\mathbb{C}$ has the wHPP.*
(ii) *Let $\mathcal{M}$ be a matroid of rank r with ground set $E = [n]$ that can be represented over $\mathbb{C}$ by an $r \times n$ matrix A for which every $r \times r$ subdeterminant is either zero or has modulus 1. Then $\mathcal{M}$ has the HPP.*

Let $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ be a class of subsets of E. A *partial transversal* of $\mathcal{A}$ is an injection ϕ from a subset $S \subseteq E$ to $\mathcal{A}$ such that $i \in \phi(i)$ for all $i \in S$. A *transversal* matroid is formed by the class of partial transversals of $\mathcal{A}$. The set of partial transversals of $\mathcal{A}$ can be identified also with the set of matchings in the bipartite graph with bipartition $V = \mathcal{A} \cup E$ and edge set $\{A_i j : j \in A_i\}$. A subset $S \subseteq E$ is called independent if it can be matched to $\mathcal{A}$. We denote $M[G,E]$ the transversal matroid on ground set E generated by the bipartite graph G. Let $\lambda_{A_i j}$ be a weight associated to each edge $A_i j$ in the bipartite graph and let:

$$a_S = \sum_{\substack{\text{matchings M} \\ V(M)\cap E = S}} \prod_{A_i j \in M} \lambda_{A_i j}$$

If we consider the graph G as the complete graph $K^{r,n}$ with the weights $\lambda_{A_i j} = 0$ if $A_i j \notin G$ then we will have that

$$a_S = \mathrm{per}(\Lambda|S), \text{ where } \Lambda \text{ is the matrix in } M_{r\times n} \text{ with } \Lambda_{i,j} = \lambda_{A_i j}$$

and where $\Lambda|S$ denotes the square submatrix of Λ using the columns indexed by the set S. The matroid $\mathcal{M}$ is called *nice* if there exists a pair (G,E) and a matrix Λ such that $\mathrm{per}(\Lambda|S)$ has the same nonzero value for all $|S| = r$ and $\mathcal{M} = M[G,E]$.

Theorem 2.13 (Permanent condition [**COSW04**] Theorem 10.2)**.** *Let Λ be an arbitrary nonnegative $r \times n$ matrix ($r \leq n$), and define P_Λ by*

$$P_\Lambda(z) = \mathrm{per}(\Lambda Z), \textit{where } Z = \mathrm{diag}(z_1, \dots, z_n)$$

Then

(i) *P_Λ is stable.*
(ii) $P_\Lambda = \sum_{\substack{S\subseteq[n] \\ |S|=r}} \mathrm{per}(\Lambda|S) z^S$.

Corollary 2.14 ([**COSW04**] Corollary 10.3)**.**

(i) *Every transversal matroid has the wHPP.*
(ii) *Every nice transversal matroid has the HPP.*

2.4. Matroid operations that preserve stability

In [**COSW04**] they described and proved that some matroid operations preserve the stability of the polynomials that represent them (i.e., $\mathrm{supp}(f) = \mathcal{B}_\mathcal{M}$).

- Taking duals.
- Taking minors (deletions and contractions).
- Direct sum.
- Parallel connection.
- 2-sum.
- Series connection.
- Full-rank matroid union.
- Some special cases of principal extension, principal truncation, principal coextension and principal cotruncation (see [**COSW04**]).

2.5. Database of matroids and its HPP-wHPP representability

We present a table that summarizes some of the results until today, concerning the *HPP* and *wHPP* for particular matroids. We are using results from [**COSW04**], [**CW06**], [**Brä07**], [**WW09**] and Chapter 3 of this work. In each case the column "Reference" points the interested reader to the source of the result where more information can be found.

Table 2.1: Some known matroids and its HPP-wHPP representability

Matroid	***HPP***	***wHPP***	**Reference**
$\forall \mathcal{M}, \lvert E \rvert \leq 6$	✓	✓	[**COSW04**]
$\forall \mathcal{M}, rank(\mathcal{M}) \leq 2$ or $rank(\mathcal{M}') \leq 2$	✓	✓	[**COSW04**]
Uniform matroids $U_{k,n}$	✓	✓	[**COSW04**]
Transversal matroids	"Nice" ✓ Not "Nice" ***unknown***	✓	[**COSW04**]
Sixth-root-of-unity matroids	✓	✓	[**COSW04**]
Binary matroids	Regular ✓ Not Regular ✗	Regular ✓ Not Regular ✗	[**COSW04**] Corollary 3.28
$PG(r-1, q)$	✗	✗	[**CW06**] Theorem 3.41
Q_7	✓	✓	[**COSW04**]
S_7	✓	✓	[**COSW04**]
F_7 (Fano)	✗	✗	[**COSW04**] [**Brä07**] Theorem 3.41 Example 3.7 Example 3.18
F_7^- (non-Fano)	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
F_7^{--}	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
F_7^{-3}	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
$M(K_4) + e$	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
F_7^{-4}	✓	✓	[**WW09**]
$W^3 + e$	✓	✓	[**WW09**]
F_7^{-5}	✓	✓	[**COSW04**]
F_7^{-6}	✓	✓	[**COSW04**]
P_7	✓	✓	[**COSW04**]
P_7'	✓	✓	[**WW09**]
P_7''	✓	✓	[**COSW04**]
P_7'''	✓	✓	[**COSW04**]
W^{3+}	✓	✓	[**WW09**]
$M(K_4)^+$	✓	✓	[**COSW04**]
$\mathcal{P}$ (Pappus)	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
$n\mathcal{P}$ (non-Pappus)	✗	***unknown***	[**COSW04**]
$n\mathcal{P} \setminus 1$	✓	✓	[**WW09**]
$n\mathcal{P} \setminus 9$	✓	✓	[**WW09**]
$(n\mathcal{P} \setminus 9) + e$	✗	✓	$\mathbb{C}$-Representable

Matroid	***HPP***	***wHPP***	**Reference**
			[**COSW04**]
P_8	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
P_8'	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
P_8''	✗	✓	$\mathbb{C}$-Representable [**COSW04**]
V_8 (Vámos)	✓	✓	[**WW09**]
S_8	✗	✗	Example 3.29 Example 3.22
$AG(3,2)$	✗	✗	Example 3.29 Example 3.8 Example 3.22
T_8	✗	✗	Example 3.21
R_9	✗	✗	Example 3.21
$PG(2,3)$	✗	✗	[**CW06**] Theorem 3.41 Example 3.18
$AG(2,3)$	✓	✓	$\sqrt[6]{1}$-matroid Proposition 3.20 Example 3.23

CHAPTER 3

Approaches to the weak half-plane property

Our main purpose will be to refine a strategy in [**Brä07**] that will allow us to reduce for some matroids the *wHPP* to the *HPP*, and also to state conditions that tell us when this reduction can be done. Using this reduction we will show, that besides F_7, there exist more matroids that are not *wHPP*-matroids. In particular, we will prove that for two families of matroids, the non-regular binary matroids and the projective geometry matroids $PG(r-1,q)$, this is also true. We also consider matroids proposed in Question 7.6 of [**COSW04**], as the coextensions $AG(3,2)$ and S_8 of F_7, and the matroids T_8 and R_9, extending the answer of [**Brä07**] to this question.

We will follow two approaches, both of them derived from the first theorem of this chapter. The first consisting in identifying the connectivity properties of a graph, and the second, in reducing the theorem to a system of equations, where we evaluate the dimension of a null space. With this second approach we provide a new proof that F_7 is not a *wHPP*-matroid. We use also the concepts and techniques developed for the Tutte-group of a matroid and valuated matroids, exposed by Dress, Wenzel and Murota in [**DW89, DW90b, DW90a, DW92, Mur96, Mur06**], to derive the main conclusions of the chapter.

We start with a result of [**Brä07**] that gives a set of restrictions to the space of coefficients where a particular matroid can have a representation with a stable polynomial.

Theorem 3.1 ([**Brä07**] Lemma 6.1)**.** *Let $f(z) = \sum_{T\subseteq[n]} a_T z^T$ be a homogeneous multi-affine polynomial with the half-plane property. Suppose that $S\cup\{i,j\} \notin$ supp(f). Then*

$$a_{S\cup\{i,k\}}a_{S\cup\{j,l\}} = a_{S\cup\{i,l\}}a_{S\cup\{j,k\}}, \textit{ for all } k,l\in[n] \textit{ with } k,l\neq i,j \qquad (3.1)$$

Proof. By Theorem 2.8 the coefficients of a homogeneous polynomial with the half-plane property all have the same phase. So we assume all coefficients are non-negative and real. By considering $(\partial^S f)_{z_r=0,\forall r\notin\{i,j,k,l\}}$, and propositions 2.3 and 2.5, we may assume without loss of generality that $S=\emptyset$ and so

$$f = a_{\{i,k\}}z_iz_k + a_{\{i,l\}}z_iz_l + a_{\{j,k\}}z_jz_k + a_{\{j,l\}}z_jz_l + a_{\{l,k\}}z_lz_k$$

Then,

$$\Delta_{ij}(f) = (a_{\{i,k\}}z_k + a_{\{i,l\}}z_l)(a_{\{j,k\}}z_k + a_{\{j,l\}}z_l)$$

We know by the Theorem 2.9 that f is stable if and only if $\Delta_{ij}(f)(x_l,x_k)\geq 0$ for all $x_l,x_k\in\mathbb{R}$ which in this case is true if and only if

$$a_{\{i,k\}}a_{\{j,l\}} = a_{\{i,l\}}a_{\{j,k\}}$$

. ∎

Clearly, Theorem 3.1 is also a necessary condition for the polynomial f with $\mathrm{supp}(f) = \mathcal{B}$ to be stable.

We introduce a definition that will play a central role for the reduction we want to acommplish.

Definition 3.2 (Degenerate quadrangle). Let $(E, \mathcal{I})$ be a finite matroid. We say that $S_1, S_2, S_3, S_4 \in \mathcal{B}$ form a *degenerate quadrangle* if

$$S_1, S_2, S_3, S_4 = S \cup \{i, k\}, S \cup \{i, l\}, S \cup \{j, k\}, S \cup \{j, l\}$$

for some set S with $i, j, k, l \notin S$, and at most one of $S \cup \{i, j\}$ and $S \cup \{k, l\}$ are bases.

In other words, for every degenerate quadrangle in a matroid $\mathcal{M}$ that has the *wHPP*, we will have one of the relations given by Theorem 3.1 for the coefficients of a stable polynomial that represents $\mathcal{M}$.

3.1. An approach using the connectivity of a graph

As a first approach we will describe, basically, the proof in [**Brä07**] for the Fano matroid (F_7) with a bit more of generality in the light of Definition 3.2.

Let $x, y \in E$ we define G_{xy} to be the graph with vertex set:

$$V_{xy} = \{T \subset E \setminus \{x, y\} : T \cup \{x\}, T \cup \{y\} \in \mathcal{B}\}$$

And edge set:

$$E_{xy} = \{TS \in V_{xy} \times V_{xy} : S \cup \{x\}, S \cup \{y\}, T \cup \{x\}, T \cup \{y\} \text{ form a degenerate quadrangle}\}$$

Lemma 3.3. *Let $x \neq y, x, y \in E$. The quotient*

$$\frac{a_{S \cup \{x\}}}{a_{S \cup \{y\}}}$$

is the same for all S in each connected component of G_{xy}.

Proof. Let $T, S \in V_{xy}$ be such that $TS \in E_{xy}$. Since $|T \Delta S| = 2$ by the definition of degenerate quadrangle, let $T \Delta S = \{i, j\}$ for some $i, j \in E,\ i \in S,\ j \in T$. By the same definition either $S \setminus \{i\} \cup \{x, y\}$ or $S \cup \{j\}$ are not in $\mathcal{B}$.

If $S \setminus \{i\} \cup \{x, y\} \notin \mathcal{B}$ then by Theorem 3.1

$$a_{S \cup \{x\}} a_{T \cup \{y\}} = a_{T \cup \{x\}} a_{S \cup \{y\}}$$

and if $S \cup \{j\} \notin \mathcal{B}$ then again:

$$a_{S \cup \{x\}} a_{T \cup \{y\}} = a_{S \cup \{y\}} a_{T \cup \{x\}}$$

both cases implies that

$$\frac{a_{S \cup \{x\}}}{a_{S \cup \{y\}}} = \frac{a_{T \cup \{x\}}}{a_{T \cup \{y\}}}.$$

□

We define for G_{xy} connected and any $T \in V_{xy}$:

$$\lambda_{xy} := \frac{a_{T \cup \{x\}}}{a_{T \cup \{y\}}}$$

Lemma 3.4. *Assume $\mathcal{M}$ to be a simple matroid such that for any triplet $\{x, y, z\} \subseteq E$*

(i) *There exist* T *such that* $T \cup \{x\}, T \cup \{y\}, T \cup \{z\} \in \mathcal{B}$*, or,*
(ii) $\{x, y, z\} \notin \mathcal{I}$ *and there exist* T *such that* $T \cup \{x, z\}, T \cup \{y, z\} \in \mathcal{B}$.

Let $x, y, z \in E$ *be distinct. If* G_{xy} *is connected for all* x, y *then*

$$\lambda_{xz} = \lambda_{xy}\lambda_{yz}$$

PROOF. If the first condition holds, then

$$\lambda_{xz} = \frac{a_{T\cup\{x\}}}{a_{T\cup\{y\}}}\frac{a_{T\cup\{y\}}}{a_{T\cup\{z\}}} = \lambda_{xy}\lambda_{yz}$$

When the second condition holds, then for any $u \in T$

$$\begin{aligned}\lambda_{xz} &= \lambda_{xu}\lambda_{uz} \\ &= \lambda_{xy}\lambda_{yu}\lambda_{uy}\lambda_{yz} \\ &= \lambda_{xy}\lambda_{yz}\end{aligned}$$

□

Lemma 3.5. *Suppose that* G_{xy} *is connected for all* x, y *and* $\lambda_{xz} = \lambda_{xy}\lambda_{yz}$ *for any distinct* $x, y, z \in E$*. Then there are positive real numbers* v_i *for all* $i \in E$*, and a complex number* C *such that*

$$a_B = C \prod_{i\in B} v_i, \ \forall B \in \mathcal{B}$$

PROOF. Let $v_i = \lambda_{i1}$, so $\lambda_{xy} = \lambda_{x1}\lambda_{1y} = v_x/v_y$ for all $x, y \in E$. Let $B_1, B_2 \in \mathcal{B}$. If $B_1 \Delta B_2 = \{j, k\}$ then

$$\frac{a_{B_1}}{a_{B_2}} = \lambda_{jk} = \frac{v_j}{v_k} = \frac{\prod_{i\in B_1} v_i}{\prod_{i\in B_2} v_i}$$

Otherwise, by the exchange axiom, there is a path

$$B_1 = A_1 \to A_2 \to \ldots \to A_p = B_2$$

such that $|A_i \Delta A_{i+1}| = 2$ and so

$$\frac{a_{B_1}}{a_{B_2}} = \frac{a_{A_1}}{a_{A_2}} \cdots \frac{a_{A_p-1}}{a_{A_p}} = \frac{\prod_{i\in B_1} v_i}{\prod_{i\in B_2} v_i}$$

Then

$$\frac{a_B}{\prod_{i\in B} v_i} = C$$

where C does not depend on B. □

Theorem 3.6. *Let* $\mathcal{M}$ *be a matroid such that* G_{xy} *is connected for all* x, y *and* $\lambda_{xz} = \lambda_{xy}\lambda_{yz}$ *for any distinct* $x, y, z \in E$*. Then wHPP=HPP.*

PROOF. Using Lemma 3.5 and after the change of variables $z_i \mapsto z_i/v_i$, the stability of the polynomial that represents $\mathcal{M}$ is preserved. So we will have that

$$\sum_{B\in\mathcal{B}} z^B$$

is stable. □

We can use Theorem 3.6 for the cases where we can find that $\lambda_{xz} = \lambda_{xy}\lambda_{yz}$ for any distinct $x, y, z \in E$. Lemma 3.4 gives two sets of conditions where this is true.

EXAMPLE 3.7. Consider the Fano matroid, F_7 of Figure 1.3a. In [**Brä07**], he proved that this matroid satisfies the second condition in Lemma 3.4 and also G_{xy} is connected for all x, y and isomorphic to the graph of Figure 3.1. Then by Theorem 3.6, *wHPP*=*HPP* for Fano. Also, since in [**COSW04**] they showed that Fano does not have the *HPP*. Hence it has neither of these properties.

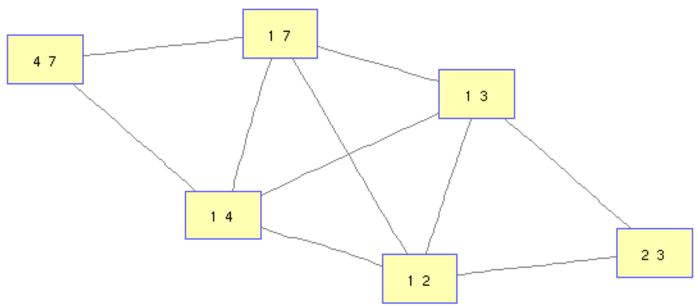


Figure 3.1: G_{56} for the Fano matroid (F_7).

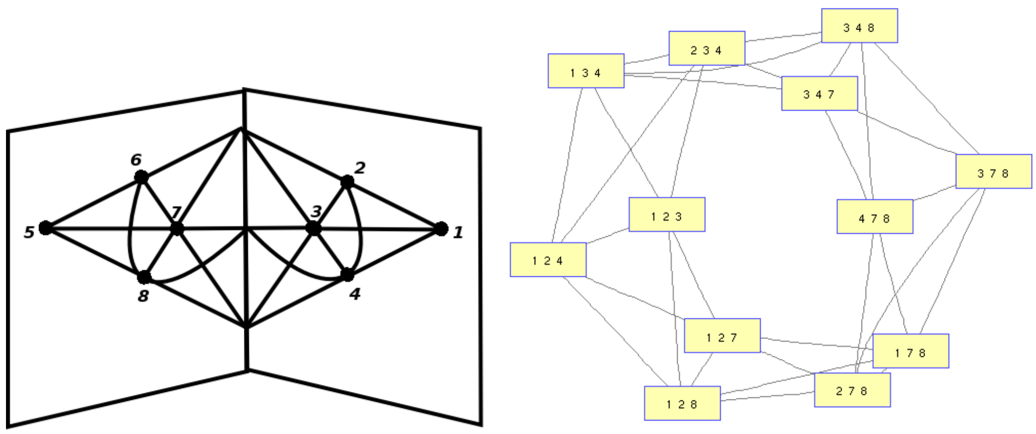
EXAMPLE 3.8. Consider the matroid $AG(3, 2)$ in Figure 3.2a, which is one of the coextensions of the Fano matroid. For this matroid G_{xy} is connected and it can be easily verified that it satisfies the first condition in Lemma 3.4. Hence *wHPP*=*HPP* for this matroid. Since the matroid is not regular, by Proposition 3.19, it is not *wHPP*.

There are some matroids that do not satisfy that G_{xy} is connected, for example $PG(2, 3)$ (Figure 3.3) and $AG(2, 3)$ (Figure 3.4). However, we will prove in the next section using a second approach that *wHPP*=*HPP* also for these matroids.

3.2. Approach using the dimension of the solution of a system of equations

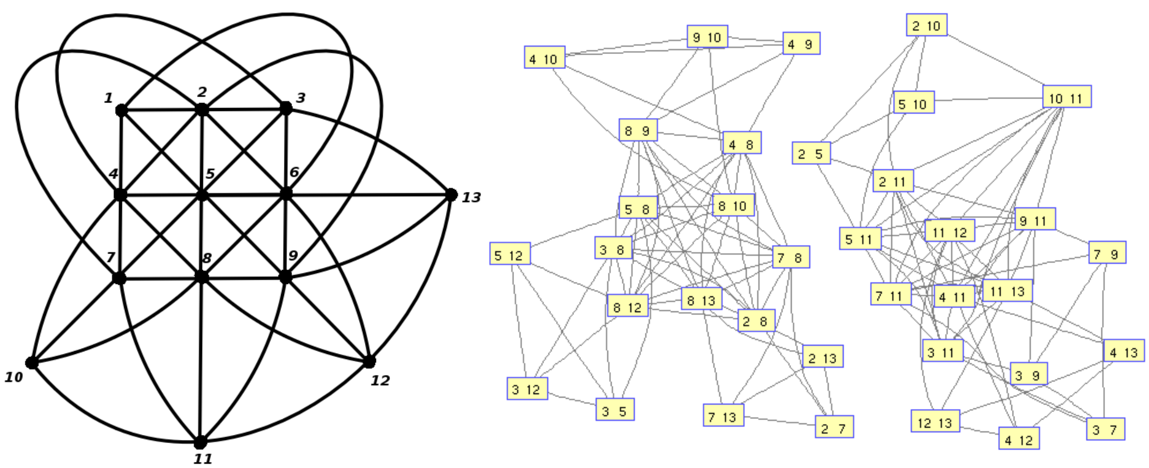
Let $\mathcal{M}$ be a matroid. And suppose that $f(z) = \sum_{B\in\mathcal{B}_\mathcal{M}} a_B z^B$ is a stable polynomial such that $\mathrm{supp}(f) = \mathcal{B}_\mathcal{M}$. As a consequence of Theorem 2.8 we can assume that all the coefficients in the support of f are real and positive. So let us assume that $a_B = e^{b_B}$ for $b_B \in \mathbb{R}$. Taking logarithms in the Theorem 3.1 ($b_B = \log(a_B)$) implies that if $S \cup \{i, j\} \notin \mathrm{supp}(f)$ and if $S \cup \{i, k\}, S \cup \{j, l\}, S \cup \{i, l\}, S \cup \{j, k\} \in \mathrm{supp}(f)$ (a degenerate quadrangle) then

$$b_{S\cup\{i,k\}} + b_{S\cup\{j,l\}} - b_{S\cup\{i,l\}} - b_{S\cup\{j,k\}} = 0, \text{ for all } k, l \in [n] \text{ with } k, l \neq i, j \quad (3.2)$$



(a) $AG(3,2)$ (b) G_{56}

Figure 3.2: G_{xy} for $AG(3,2)$.

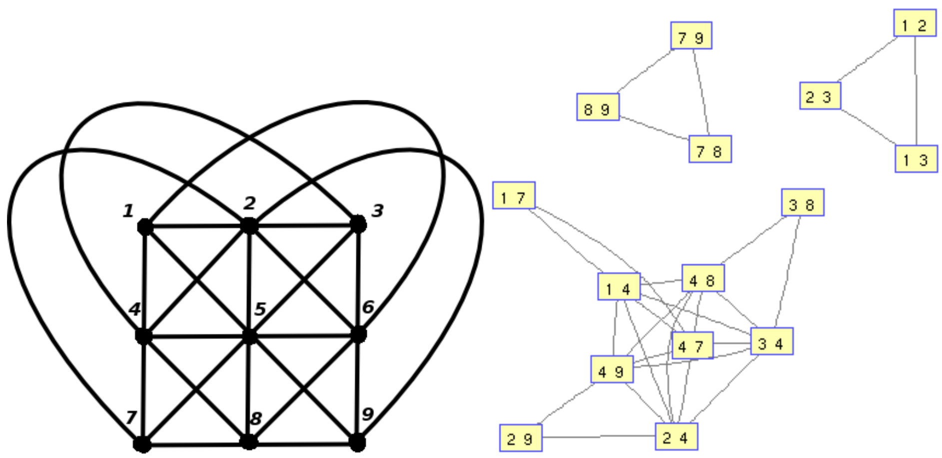


(a) $PG(2,3)$ (b) G_{16}

Figure 3.3: G_{xy} for $PG(2,3)$ is disconnected.

We get a system of equations of the form $A_{\mathcal{M}}X = 0$ where X is the vector of variables b_T, $T \in \mathcal{B}$. This system has a non-empty set of non-zero solutions given by the following lemma:

Lemma 3.9. *Let $v_1, v_2, \ldots, v_n$ be a set of $n = |E|$ real numbers. Then $b_B = \sum_{t \in B} v_t$, for all $B \in \mathcal{B}$, is a solution to System 3.2.*



(a) $AG(2,3)$ (b) G_{56}

Figure 3.4: G_{xy} for $AG(2,3)$ is disconnected.

Proof. Plug a solution of this form in the left-hand side of Equation 3.2. Then

$$\sum_{t\in S} v_t + v_i + v_k + \sum_{t\in S} v_t + v_j + v_l - \sum_{t\in S} v_t - v_i - v_l - \sum_{t\in S} v_t - v_j - v_k = 0.$$

□

We want to study the space of values for the coefficients of f that satisfy the necessary condition in Theorem 3.1 using the properties of the null space of the matrix $A_{\mathcal{M}}$ ($N(A_{\mathcal{M}})$). In particular, we will show that the dimension of $N(A_{\mathcal{M}})$ is useful to determine when *wHPP=HPP*. If we can find that the dimension of $N(A_{\mathcal{M}})$ is equal to the dimension of the vector space spanned by the solution given by Lemma 3.9, then *wHPP=HPP* for $\mathcal{M}$.

Theorem 3.10. *For a matroid $\mathcal{M}$. Let $\mathcal{B} = \{B_1, \dots, B_m\}$ and suppose that*

$$\dim(N(A_{\mathcal{M}})) = \dim(\{(\sum_{t\in B_1} v_t, \dots, \sum_{t\in B_m} v_t) : v_t \in \mathbb{R} \textit{ for all } t \in E\}).$$

Then wHPP=HPP for $\mathcal{M}$.

Proof. Let $V = \{(\sum_{t\in B_1} v_t, \dots, \sum_{t\in B_m} v_t) : v_t \in \mathbb{R} \text{ for all } t \in E\}$. By Lemma 3.9, $V \subseteq N(A_{\mathcal{M}})$. If $\dim(V) = \dim(N(A_{\mathcal{M}}))$, then in fact $V = N(A_{\mathcal{M}})$, so any solution for System 3.2 is of the form $b_B = \sum_{t\in B} v_t$, for all $B \in \mathcal{B}$. Hence $a_B = e^{\sum_{t\in B} v_t}$. Do the change of variable $z_t \mapsto z_t/e^{v_t}$. This change of variable preserves the stability of the polynomial and the support, but the coefficients of the new polynomial are zeros and ones. □

Since the dimension of the solution in Lemma 3.9 is at most n, we immediately know that if $\dim(N(A_{\mathcal{M}})) > n$, we cannot apply Theorem 3.10 for $\mathcal{M}$. This is a good start to identify candidates for this reduction.

Table 3.1 shows some matroids for which $\dim(N(A_{\mathcal{M}}))$ has been computed. The matroids for which $\dim(N(A_{\mathcal{M}})) = n$ are highlighted. The computations to determine the connectivity of the graph in the last section and to calculate the dimension of the null space have been done with the aid of MATLAB.

Table 3.1: Dimension of $N(A_{\mathcal{M}})$ for some matroids

Matroid	$n = \lvert E \rvert$	$\dim(N(A_{\mathcal{M}}))$	$\lvert \mathcal{B} \rvert$
$M(K_4)$	6	6	16
W^3	6	8	17
F_7 (Fano)	7	7	28
F_7^- (non-Fano)	7	8	29
F_7^{--}	7	10	30
F_7^{-3}	7	13	31
F_7^{-4}	7	17	32
F_7^{-5}	7	22	33
F_7^{-6}	7	28	34
$U_{3,7}$	7	35	35
$M(K_4) + e$	7	13	31
$W^3 + e$	7	17	32
V_8	8	18	63
W^4	8	24	52
S_8	8	8	48
T_8	8	8	59
$AG(3,2)$	8	8	56
$AG(3,2)'$	8	9	57
R_8	8	10	58
F_8	8	10	58
Q_8	8	11	59
L_8	8	17	62
$AG(2,3)$	9	9	72
R_9	9	9	69
Pappus	9	16	75
$n\mathcal{P}$ (non-Pappus)	9	17	76
Non-Desargues	10	27	111
$PG(2,3)$	13	13	234

The task that we have in our hands is to determine if the solution given in Lemma 3.9 has dimension n for the matroids with $\dim(N(A_{\mathcal{M}})) = n$ in Table 3.1. The following results will help us finding an answer for this.

Lemma 3.11. *Let $\mathcal{M}$ be a matroid. For a nonempty subset $S \subseteq E$ the following are equivalent:*

(i) *S is a connected component.*
(ii) *S is maximal with respect to the property that for any pair $i, j \in S$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$.*

Proof. Since the lemma for the case of S being a loop is trivial, it is enough to prove that for any $i, j \in E$ there exists a circuit C containing i, j if and only if there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$. So assume first that for i, j there exists a circuit $C \supseteq \{i, j\}$. Since $C \setminus \{i\}$ is independent, there is a basis B such that $C \setminus \{i\} \subseteq B$. Now, $B \cup \{i\}$ contains a unique circuit $C(i, B)$ and since $C \subseteq B \cup \{i\}$, in fact, $C(i, B) = C$. However, $B \setminus \{j\} \cup \{i\}$ contains no circuit. Hence it is a basis and $B \Delta (B \setminus \{j\} \cup \{i\}) = \{i, j\}$.
Now assume that for i, j there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$. Assume without loss of generality that $B_1 = T \cup \{i\}$ and $B_2 = T \cup \{j\}$ for some $T \subseteq E$. Since $j \in C(j, B_1)$ and $i \in C(i, B_2)$, and $B_1 \cup \{j\} = B_2 \cup \{i\} = T \cup \{i, j\}$ then, by the uniqueness of $C(j, B_1)$ and $C(i, B_2)$, we have $C(j, B_1) = C(i, B_2)$. The maximality part comes from the definition of connected component. □

Lemma 3.12. *For a matroid $\mathcal{M}$ the following are equivalent:*

(i) *$\mathcal{M}$ is connected.*
(ii) *For any pair $i, j \in E$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$.*

Proof. If $\mathcal{M}$ is connected by Lemma 3.11 it follows that for any pair $i, j \in E$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$.
Assume now that for any pair $i, j \in E$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \Delta B_2 = \{i, j\}$. Then by Lemma 3.11 E is a connected component of $\mathcal{M}$ and hence $\mathcal{M}$ is connected. □

Lemma 3.13. *Let $\mathcal{M}$ be a matroid with z connected components. Then the dimension of the solution given by Lemma 3.9 is $n - z + 1$.*

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^m$, with $m = |\mathcal{B}|$, be the linear operator that maps

$$(v_1, v_2, \ldots, v_n) \longmapsto (\sum_{t \in B_1} v_t, \sum_{t \in B_2} v_t, \ldots, \sum_{t \in B_m} v_t)$$

Clearly the dimension of $T(\mathbb{R}^n)$ is at most n. To identify the null space of T let us assume that $\sum_{t \in B} v_t = 0$ for all $B \in \mathcal{B}$. If we take any $i, j \subseteq E$ that belong to the same non-loop connected component of $\mathcal{M}$, then by Lemma 3.11, there exist bases B_1 and B_2 such that $B_1 \Delta B_2 = \{i, j\}$. We can assume without loss of generality that $B_1 = S \cup \{i\}$ and $B_2 = S \cup \{j\}$ for some set $S \subset E$. Since $\sum_{t \in S} v_t + v_i = \sum_{t \in B_1} v_t = 0$ and $\sum_{t \in S} v_t + v_j = \sum_{t \in B_2} v_t = 0$, we have that $v_i = v_j$. This implies that the v_i are the same for any connected component in $\mathcal{M}$. If we write the matroid as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_z$ on disjoint sets $E_{\mathcal{M}} = E_{\mathcal{M}_1} \uplus E_{\mathcal{M}_2} \uplus \ldots \uplus E_{\mathcal{M}_z}$ and using the equation $\sum_{t \in B} v_t = 0$ for any $B \in \mathcal{B}$, we can express that all the v_i in the kernel must satisfy the relation:

$$\operatorname{rank}(\mathcal{M}_1)[v]_1 + \ldots + \operatorname{rank}(\mathcal{M}_z)[v]_z = 0$$

where $[v]_j$ denotes the common value of v_i for $i \in E_{\mathcal{M}_j}$. Then the null space of T is of dimension $z - 1$ and the dimension of $T(\mathbb{R}^n)$ is $n - z + 1$. □

Any connected component of a matroid is its minor. Since taking minors and the operation of direct sum preserve stability, normally it is enough to analyze only connected components to determine the *wHPP*-representability of a matroid.

Corollary 3.14. *Let $\mathcal{M}$ be a connected matroid. Then the dimension of the solution given by Lemma 3.9 is n.*

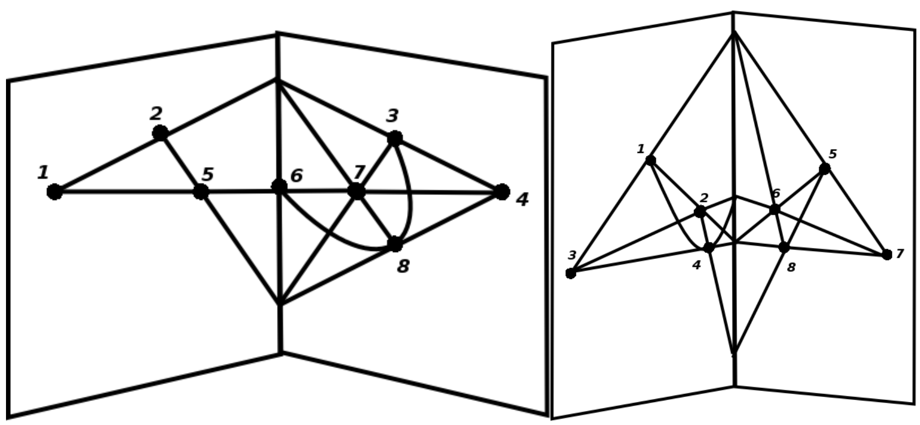
Corollary 3.15. *Let $\mathcal{M}$ be a matroid generated by a projective geometry $PG(r-1,q)$. Then the solution of Lemma 3.9 has dimension n.*

PROOF. $PG(r-1,q)$ is connected. □

Corollary 3.16. *Let $\mathcal{M}$ be a matroid generated by an affine geometry $AG(r-1,q)$. Then the solution of Lemma 3.9 has dimension n.*

PROOF. $AG(r-1,q)$ is connected. □

It is easy to verify that the matroids T_8 (Figure 3.5b), S_8 (Figure 3.5a), R_9 (Figure 3.6a) and $M(K_4)$ (Figure 3.6b) are also connected, and so, also the solution in Lemma 3.9 for these matroids has dimension n.



(a) S_8 (b) T_8

Figure 3.5: Matroids S_8 and T_8.

For the next example (3.18) let us state this result from [**CW06**]:

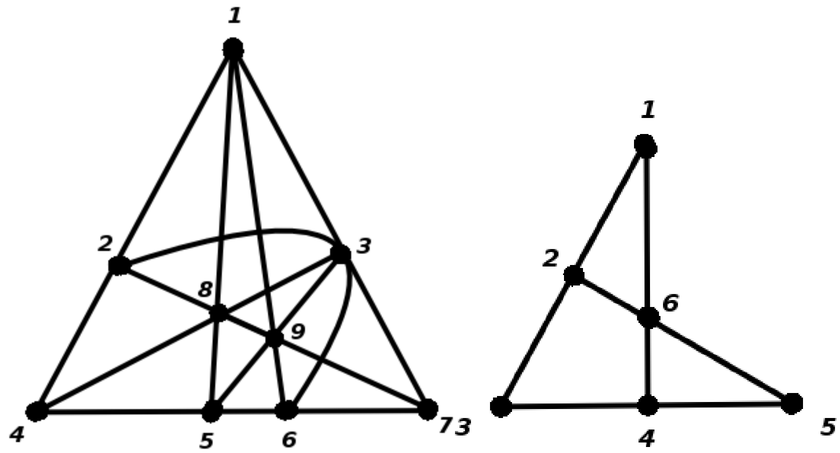
Proposition 3.17 ([**CW06**] Proposition 5.8)**.** *No finite projective geometry is a HPP matroid.*

EXAMPLE 3.18. In table 3.1 we have $\dim(N(A_{\mathcal{M}})) = n$ for the matroids F_7 ($PG(2,2)$) and $PG(2,3)$. By Corollary 3.15 and Theorem 3.10 this implies that *wHPP*=*HPP* for these matroids. By Proposition 3.17 then we have that neither F_7 nor $PG(2,3)$ have the *wHPP*. This is a new proof of the fact that F_7 is not a *wHPP*-matroid.

The following results in [**COSW04**] will be used in examples 3.21, 3.22 and 3.23.

Proposition 3.19 ([**COSW04**] Corollary 8.16)**.** *A binary matroid has the HPP if and only if it is regular.*

Proposition 3.20 ([**COSW04**] Corollary 8.17)**.** *A ternary matroid has the HPP if and only if it is a $\sqrt[6]{1}$-matroid.*



(a) R_9 (b) $M(K_4)$

Figure 3.6: Matroids R_9 and $M(K_4)$.

Example 3.21. The matroids T_8 (Figure 3.5b) and R_9 (Figure 3.6a) are representable only over fields of characteristic 3. Since $\dim(N(A_{\mathcal{M}}))$ and the dimension of the solution of Lemma 3.9 are n, Theorem 3.10 implies that *wHPP=HPP* for these matroids. Since these matroids are not representable over $\mathbb{C}$, in particular, they are not $\sqrt[6]{1}$-matroids. Proposition 3.20 implies that they are not *wHPP*.

Example 3.22. The matroids $AG(3,2)$ (Figure 3.2a) and S_8 (Figure 3.5a) are coextensions of F_7 and they are only representable over fields of characteristic 2. Again, by Theorem 3.10, for these matroids *wHPP=HPP*. Since they are not regular, Proposition 3.19 implies that they are not *wHPP*. A different case is the matroid $M(K_4)$ which is regular, here also *wHPP=HPP*. $M(K_4)$ is *wHPP* (something already known since this matroid has a ground set with 6 elements).

Example 3.23. The matroid $AG(2,3)$ is a ternary matroid and also a $\sqrt[6]{1}$-matroid. By Corollary 3.16 and since $\dim(N(A_{\mathcal{M}})) = n$, using Theorem 3.10 we conclude *wHPP=HPP*. By Proposition 3.20 $AG(2,3)$ has the *wHPP*.

3.3. Using valuations

In Section 1.3 we introduced the concept of valuated matroids. We want to show for a particular matroid $\mathcal{M}$ represented by a stable polynomial f, that the function given by $v(B) = \log(a_B)$ is a valuation. We can do this by verifying a local condition, that Inequality 1.2 holds. If it is true that v is a valuation for $\mathcal{M}$ and also that $\mathcal{M}$ is rigid it follows that *wHPP=HPP*. We apply this method to the family of binary matroids.

Lemma 3.24. *Let $\mathcal{M}$ be a binary matroid. Suppose $f(z) = \sum_{B\in\mathcal{B}_{\mathcal{M}}} a_B z^B$ is a stable polynomial such that* $\operatorname{supp}(f) = \mathcal{B}_{\mathcal{M}}$. *Then the function $v(B) = \log(a_B)$ is a valuation of $\mathcal{M}$.*

Proof. According to Theorem 1.9 we just need to prove that the local exchange condition for v is satisfied. So let $B_1, B_2 \in \mathcal{B}$ with $|B_1 \Delta B_2| = 4$. Since both B_1, B_2 are bases, this implies without loss of generality that $B_1 = S \cup \{a, b\}$ and $B_2 = S \cup \{c, d\}$ for some $\{a, b, c, d\} = B_1 \Delta B_2$ and $S \subset E$, $|S| = r - 2$. Since $\mathcal{M}$ is binary $S \cup \{s, t\}$ is a circuit-hyperplane for some $\{s, t\} \in \binom{\{a,b,c,d\}}{2}$. If not, for all $\{s, t\} \in \binom{\{a,b,c,d\}}{2}$ it happens that $S \cup \{s, t\} \in \mathcal{B}$, and then the matroid $(\mathcal{M}^{/S})^{\setminus(E \setminus \{a,b,c,d\})}$ is a minor of $\mathcal{M}$ isomorphic to $U_{2,4}$, contradicting that $\mathcal{M}$ is binary. So assume without loss of generality that $S \cup \{a, c\} \notin \mathcal{B}$ (it is a circuit hyperplane) and let in Theorem 3.1 $i = a, j = c, k = b$ and $l = d$. Then we get:

$$a_{S\cup\{a,b\}} a_{S\cup\{c,d\}} = a_{S\cup\{a,d\}} a_{S\cup\{b,c\}}$$

and so

$$\log(a_{S\cup\{a,b\}}) + \log(a_{S\cup\{c,d\}}) = \log(a_{S\cup\{a,d\}}) + \log(a_{S\cup\{b,c\}})$$
$$v(a_{S\cup\{a,b\}}) + v(a_{S\cup\{c,d\}}) = v(a_{S\cup\{a,d\}}) + v(a_{S\cup\{b,c\}})$$

So the function v satisfies the local exchange axiom and hence it is a valuation of $\mathcal{M}$. □

Theorem 3.25 ([**DW92**] Theorem 5.11)**.**

(i) *Every binary matroid is rigid.*
(ii) *Every finite projective space of dimension at least two is rigid.*

Corollary 3.26. *For any binary matroid all the solutions to the system of equations 3.2 have the form* $b_B = \sum_{i \in B} g(i)$ *for a function* $g : E \to \mathbb{R}$.

Proof. According to Lemma 3.24, $b_B = \log(a_B) = v(B)$ is a valuation. Since every binary matroid is rigid then the valuation must have this form and hence also b_B. □

Lemma 3.27. *For any binary matroid wHPP=HPP.*

Proof. Since by Corollary 3.26, $b_B = \sum_{i \in B} g(i)$ for a function $g : E \to \mathbb{R}$, then $a_B = e^{\sum_{i \in B} g(i)}$. Do the change of variable $z_i \mapsto z_i / e^{g(i)}$. This change of variable preserves the stability of the polynomial and the support, but the coefficients of the new polynomial are zeros and ones. □

Using Proposition 3.19 we get to the conclusion:

Corollary 3.28. *A binary matroid has the wHPP property if and only if it is regular.*

Example 3.29. Consider S_8 (Figure 3.5a) and $AG(3, 2)$ (Figure 3.2a) that are coextensions of F_7. Since these are only representable over a field of characteristic 2, by Corollary 3.28 we know that they do not have the *wHPP*.

3.4. Using the Tutte group

In [**DW89**], Dress and Wenzel introduced the concept of the Tutte-group of a matroid $\mathcal{M}$ and some other related groups in order to study matroids with coefficients. This work was also used by them when working with the projective equivalence of valuated matroids and proving the results we have used in the previous section. We will relate algebraically the set of coefficients of a stable polynomial with one of these Tutte-groups. This will permit us to prove that if the inner Tutte-group for $\mathcal{M}$ is a torsion group then $\dim(N(A_{\mathcal{M}})) = n-z+1$ where z is the number of connected components of $\mathcal{M}$. We use this method to show that no projective geometry is a *wHPP*-matroid.

In the following definitions, let $\mathcal{M}$ be a matroid of rank r.

DEFINITION 3.30. Let $\mathbb{F}_{\mathcal{M}}$ denote the free abelian group generated by the symbols ε and X_{B_1,B_2} for $B_1, B_2 \in \mathcal{B}$ and $|B_1 \cap B_2| = r-1$ and let $\mathbb{K}_{\mathcal{M}}$ be the subgroup of $\mathbb{F}_{\mathcal{M}}$ generated by all the elements of the form:

(1) ε^2.
(2) $X_{B_1,B_2}X_{B_2,B_1}$ where $B_1, B_2 \in \mathcal{B}$ and $|B_1 \cap B_2| = r-1$.
(3) $X_{B_1,B_2}X_{B_2,B_3}X_{B_3,B_1}$ where $B_1, B_2, B_3 \in \mathcal{B}$ and $|B_1 \cap B_2 \cap B_3| = r-1$.
(4) $\varepsilon X_{B_1,B_2}X_{B_2,B_3}X_{B_3,B_1}$ where $B_1, B_2, B_3 \in \mathcal{B}$, $|B_i \cap B_j| = r-1$ for $\{i,j\} \in \binom{[3]}{2}$ and $|B_1 \cap B_2 \cap B_3| = r-2$.
(5) $X_{B_1,B_2}X_{B_3,B_4}$ if B_1, B_2, B_3, B_4 form a degenerate quadrangle.

Then the *Tutte-group* $\mathbb{T}_{\mathcal{M}}$ of $\mathcal{M}$ is defined

$$\mathbb{T}_{\mathcal{M}} := \mathbb{F}_{\mathcal{M}}/\mathbb{K}_{\mathcal{M}}$$

DEFINITION 3.31. Let $\mathbb{F}^{\mathcal{B}}_{\mathcal{M}}$ denote the free abelian group generated by the symbols ε and $X_{(B)}$ where (B) is an ordered tuple with base set $B \in \mathcal{B}$ and let $\mathbb{K}^{\mathcal{B}}_{\mathcal{M}}$ be the subgroup of $\mathbb{F}^{\mathcal{B}}_{\mathcal{M}}$ generated by all the elements of the form:

(1) ε^2.
(2) $\varepsilon X_{(B)}X^{-1}_{\tau((B))}$ where $B \in \mathcal{B}$ and τ is an odd permutation in $\mathfrak{S}_r$.
(3) $X_{(s_1,\dots,s_{r-2},i,k)}X_{(s_1,\dots,s_{r-2},j,l)}X^{-1}_{(s_1,\dots,s_{r-2},i,l)}X^{-1}_{(s_1,\dots,s_{r-2},j,k)}$ if $\{s_1,\dots,s_{r-2},x,y\} \in \mathcal{B}$ for $\{x,y\} \in \{\{i,k\},\{j,l\},\{i,l\},\{j,k\}\}$ but $\{s_1,\dots,s_{r-2},i,j\} \notin \mathcal{B}$.

Then define the *Tutte-group based on* $\mathcal{B}$, $\mathbb{T}^{\mathcal{B}}_{\mathcal{M}}$, as:

$$\mathbb{T}^{\mathcal{B}}_{\mathcal{M}} := \mathbb{F}^{\mathcal{B}}_{\mathcal{M}}/\mathbb{K}^{\mathcal{B}}_{\mathcal{M}}$$

DEFINITION 3.32. Let $\mathbb{F}^{\mathcal{H}}_{\mathcal{M}}$ denote the free abelian group generated by the symbols ε and $X_{H,a}$ for $H \in \mathcal{H}$ and $a \in E \setminus H$, and let $\mathbb{K}^{\mathcal{H}}_{\mathcal{M}}$ be the subgroup of $\mathbb{F}^{\mathcal{H}}_{\mathcal{M}}$ generated by all the elements of the form:

(1) ε^2.
(2) $\varepsilon X_{H_1,a_2}X^{-1}_{H_1,a_3}X_{H_2,a_3}X^{-1}_{H_2,a_1}X_{H_3,a_1}X^{-1}_{H_3,a_2}$ for $H_1, H_2, H_3 \in \mathcal{H}$, $L = H_1 \cap H_2 \cap H_3 = H_i \cap H_j$ for $i \neq j$, $rank(L) = r-2$ and $a_i \in H_i \setminus L$ for all $i \in [3]$.

Then the *extended Tutte-group* $\mathbb{T}^{\mathcal{H}}_{\mathcal{M}}$ of $\mathcal{M}$ is defined as:

$$\mathbb{T}^{\mathcal{H}}_{\mathcal{M}} := \mathbb{F}^{\mathcal{H}}_{\mathcal{M}}/\mathbb{K}^{\mathcal{H}}_{\mathcal{M}}$$

DEFINITION 3.33. Let $\varphi : \mathbb{T}^{\mathcal{H}}_{\mathcal{M}} \to \mathbb{Z}^{\mathcal{H}} \otimes \mathbb{Z}^E$ the homomorphism defined by

$$\varphi([\varepsilon]) := 0 \text{ where } [\varepsilon] \text{ denotes the equivalence class of } \varepsilon \text{ in } \mathbb{T}^{\mathcal{H}}_{\mathcal{M}}.$$
$$\varphi([X_{H,a}]) := (\delta_H, \delta_a) \text{ for } H \in \mathcal{H},\ a \in E \setminus H.$$

Then the *inner Tutte-group* $\mathbb{T}^{(0)}_{\mathcal{M}}$ of $\mathcal{M}$ is defined by

$$\mathbb{T}^{(0)}_{\mathcal{M}} := \ker(\varphi)$$

From the definitions it follows that $\mathbb{T}^{(0)}_{\mathcal{M}} \subseteq \mathbb{T}_{\mathcal{M}} \subseteq \mathbb{T}^{\mathcal{H}}_{\mathcal{M}}$. The interested reader can find a treaty about Tutte-groups in [**DW89**] and also the following results:

Proposition 3.34 ([**DW89**]). *Let $\mathcal{M}$ be a matroid of rank r in a ground set $E = [n]$ with z connected components. Then*

$$\mathbb{T}_{\mathcal{M}} \cong \mathbb{T}^{(0)}_{\mathcal{M}} \otimes \mathbb{Z}^{n-z}. \tag{3.3}$$

$$\mathbb{T}^{\mathcal{B}}_{\mathcal{M}} \cong \mathbb{T}_{\mathcal{M}} \otimes \mathbb{Z}. \tag{3.4}$$

Corollary 3.35. $\mathbb{T}^{\mathcal{B}}_{\mathcal{M}} \cong \mathbb{T}^{(0)}_{\mathcal{M}} \otimes \mathbb{Z}^{n-z+1}$.

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a multiaffine homogeneous stable polynomial that represents the matroid $\mathcal{M}$. And let $\Gamma_{\mathcal{M}}$ be the subgroup of $(\mathbb{R}^+ \setminus \{0\}, \times)$ generated by the coefficients $a_B \in \mathbb{R}^+ \setminus \{0\}$, $B \in \mathcal{B}$.

Lemma 3.36. *Let $\varphi : \mathbb{F}^{\mathcal{B}}_{\mathcal{M}} \to \Gamma_{\mathcal{M}}$ be the epimorphism defined by*

$$\varphi(X_{(B)}) := a_B.$$
$$\varphi(\varepsilon) := 1.$$

Then φ induces an epimorphism:

$$\tilde{\varphi} : \mathbb{T}^{\mathcal{B}}_{\mathcal{M}} \to \Gamma_{\mathcal{M}}$$

Proof. We have first that

$$\varphi(\varepsilon^2) = \varphi(\varepsilon)^2 = 1$$

Since $X_{\tau((B))} \mapsto a_B$ for any permutation $\tau \in \mathfrak{S}_r$ then

$$\varphi(\varepsilon X_{(B)} X^{-1}_{\tau((B))}) = \varphi(\varepsilon)\varphi(X_{(B)})\varphi(X^{-1}_{\tau((B))}) = 1 a_B a_B^{-1} = 1$$

and because the condition in Theorem 3.1.

$$\varphi(X_{(s_1,\ldots,s_{r-2},i,k)} X_{(s_1,\ldots,s_{r-2},j,l)} X^{-1}_{(s_1,\ldots,s_{r-2},i,l)} X^{-1}_{(s_1,\ldots,s_{r-2},j,k)}) =$$
$$a_{\{s_1,\ldots,s_{r-2},i,k\}} a_{\{s_1,\ldots,s_{r-2},j,l\}} a^{-1}_{\{s_1,\ldots,s_{r-2},i,l\}} a^{-1}_{\{s_1,\ldots,s_{r-2},j,k\}} = 1$$

if $\{s_1, \ldots, s_{r-2}, x, y\} \in \mathcal{B}$ for $\{x, y\} \in \{\{i, k\}, \{j, l\}, \{i, l\}, \{j, k\}\}$ but $\{s_1, \ldots, s_{r-2}, i, j\} \notin \mathcal{B}$.

These three conditions above imply by Definition 3.31, that $\mathbb{K}^{\mathcal{B}}_{\mathcal{M}} \subseteq \ker(\varphi)$. and hence φ induces an epimorphism $\tilde{\varphi} : \mathbb{F}^{\mathcal{B}}_{\mathcal{M}} / \mathbb{K}^{\mathcal{B}}_{\mathcal{M}} \to \Gamma_{\mathcal{M}}/\{1\}$ (we direct the interested reader to Chapter 1, Corollary 5.8 in [**Hun80**] to verify this). And again by the definition of the Tutte-group based on $\mathcal{B}$ the lemma follows. □

Lemma 3.37. *Let $\mathcal{M}$ be a matroid with z connected components. If $\mathbb{T}^{(0)}_{\mathcal{M}}$ is a torsion group then* $\dim(N(A_{\mathcal{M}})) = n - z + 1$.

Proof. By Lemma 3.36 there exist an epimorphism $\tilde{\varphi} : \mathbb{T}^{\mathcal{B}}_{\mathcal{M}} \to \Gamma_{\mathcal{M}}$ and by Corollary 3.35 we know that $\mathbb{T}^{\mathcal{B}}_{\mathcal{M}} \cong \mathbb{T}^{(0)}_{\mathcal{M}} \otimes \mathbb{Z}^{n-z+1}$. Note that $\mathbb{Z}^{n-z+1}$ will have naturally an additive notation, so let us use here an additive notation for $\mathbb{T}^{\mathcal{B}}_{\mathcal{M}}$. Since $\mathbb{Z}$ is cyclic and has one generator (1), we have the following set of generators of $\mathbb{T}_{\mathcal{M}}^{\mathcal{B}}$:

$$\begin{gathered}(g,0,0,\dots,0)\\(0,1,0,\dots,0)\\\vdots\\(0,0,0,\dots,1)\end{gathered}$$

for every $g \in \mathbb{T}_{\mathcal{M}}^{(0)}$. For a particular g call G_g the cyclic group generated by $(g,0,0,\dots,0)$ then G_g is a cyclic torsion group and $\tilde{\varphi}(G_g)$ must be also a torsion subgroup of $(\mathbb{R}^+ \setminus \{0\}, \times)$, and since this is torsion-free, then $\tilde{\varphi}(G_g) = \{1\}$. And this happens for all $g \in \mathbb{T}_{\mathcal{M}}^{(0)}$. Then the image $\tilde{\varphi}(\mathbb{T}_{\mathcal{M}}^{\mathcal{B}})$ is at most determined by the image of the $n-z+1$ generators:

$$\begin{gathered}\tilde{\varphi}((0,1,0,\dots,0))\\\vdots\\\tilde{\varphi}((0,0,0,\dots,1))\end{gathered}$$

and hence also the coefficients a_B and by taking logarithms also $N(A_{\mathcal{M}})$. By Lemma 3.13 in fact, there is equality. $\square$

Theorem 3.38. *Let $\mathcal{M}$ be a matroid. If $\mathbb{T}_{\mathcal{M}}^{(0)}$ is a torsion group then wHPP=HPP.*

Proof. Using lemmas 3.13 and 3.37 we conclude that:

$$\dim(N(A_{\mathcal{M}})) = \dim(\{(\sum_{t\in B_1} v_t, \dots, \sum_{t\in B_m} v_t) : v_t \in \mathbb{R}\ \forall t \in E,\ m = |\mathcal{B}|\}) = n - z + 1.$$

By Theorem 3.10 this implies that *wHPP=HPP*. $\square$

Dress and Wenzel have proved that the inner Tutte-group is a torsion group for a projective geometry.

Proposition 3.39 ([**DW90a**] Theorem 3.6)**.** *For $\mathcal{M} = PG(r-1,q)$, $\mathbb{T}_{\mathcal{M}}^{(0)} \cong GF(q) \setminus \{0\}$.*

Corollary 3.40. *For $PG(r-1,q)$ wHPP=HPP.*

Proof. Since for $\mathcal{M} = PG(r-1,q)$, $\mathbb{T}_{\mathcal{M}}^{(0)} \cong GF(q) \setminus \{0\}$ is a torsion group, by Theorem 3.38 the corollary follows. $\square$

Theorem 3.41. *$PG(r-1,q)$ is not a wHPP-matroid.*

Proof. By Lemma 3.40 *wHPP=HPP* for $PG(r-1,q)$. The theorem now follows from Proposition 3.17. $\square$

This result also gives another proof for the fact that the Fano matroid F_7 is not a *wHPP* matroid. This method can be used in other cases, in particular, it is true also for a binary matroid that the inner Tutte-group is a torsion group:

Proposition 3.42 ([**Wen89**] Theorem 5.2)**.** *If $\mathcal{M}$ is binary then*

$$\mathbb{T}_{\mathcal{M}}^{(0)} \cong \begin{cases} \{0\} & \text{if the Fano matroid or its dual is a minor of } \mathcal{M} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

3.5. Open questions and other related results

We define the *i-neighborhood* of $B \subset \mathcal{B}$, $i = 1, \ldots, |B|$, to be the set $N_B^i = \{B' \subseteq \mathcal{B} : |B \Delta B'| = 2i\}$ and we will say that the i-neighborhood is *complete* if $N_B^i \subseteq \mathcal{I}$.

Lemma 3.43. *If $\mathcal{M}'$ is a relaxation of $\mathcal{M}$ then*

$$\dim(N(A_{\mathcal{M}'})) \geq \dim(N(A_{\mathcal{M}})) + 1$$

.

Proof. By the definition of relaxation and Proposition 1.5, for $\mathcal{M}'$ to be a relaxation of $\mathcal{M}$ implies that there exists a set $X \notin \mathcal{M}$ such that X is a circuit and a hyperplane and such that $\mathcal{B}_{\mathcal{M}'} = \mathcal{B}_{\mathcal{M}} \cup X$. Since X is a hyperplane it implies that for any $y \notin X$, $\text{rank}(X \cup \{y\}) = \text{rank}(\mathcal{M})$ and since it is a circuit $X \setminus \{x\} \cup \{y\}$ is independent for all $x \in X$ and hence it is a basis, this implies that the 1-neighborhood of X is complete. But this implies according Theorem 3.1 and Equation 3.2 that the variable b_X does not appear in any equation of the system $A_{\mathcal{M}'}X = 0$ also any equation between four bases that were related only by $X \notin \mathcal{B}$ will not be in this system so $\text{rank}(A_{\mathcal{M}'}) \leq \text{rank}(A_{\mathcal{M}})$, where "rank" refers to rank of a matrix . But also the number of columns increases in one since $X \in \mathcal{B}_{\mathcal{M}'}$. So then

$$\begin{aligned}
\dim(N(A_{\mathcal{M}'})) &= |\mathcal{B}_{\mathcal{M}'}| - \text{rank}(A_{\mathcal{M}'}) \\
&= |\mathcal{B}_{\mathcal{M}}| + 1 - \text{rank}(A_{\mathcal{M}'}) \\
&\geq |\mathcal{B}_{\mathcal{M}}| + 1 - \text{rank}(A_{\mathcal{M}}) \\
&= \dim(N(A_{\mathcal{M}})) + 1
\end{aligned}$$

□

Conjecture 3.44. *Let $\mathcal{M}$ be a matroid that has the wHPP. If $\mathcal{M}'$ is a relaxation of $\mathcal{M}$ then $\mathcal{M}'$ has the wHPP.*

If this conjecture is proven to be true then, for example, the non-Pappus matroid will have the *wHPP* since Pappus has. In general, it is interesting to study how is the relation between the dimension of $N(A_{\mathcal{M}})$ and the weak half-plane property. We already know that if $\dim(N(A_{\mathcal{M}})) = n - z + 1$, then *wHPP=HPP*. But what does happen when $\dim(N(A_{\mathcal{M}})) > n - z + 1$? Is $\mathcal{M}$ always a *wHPP*-matroid? or we can formulate the question:

Question 3.45. *Is there any function $g(n, z, r, m)$ such that $\mathcal{M}$ is a wHPP-matroid whenever $\dim(N(A_{\mathcal{M}})) > g(n, z, r, m)$?. Here $n = |E|$, z is the number of connected components, $r = \text{rank}(\mathcal{M})$ and $m = |\mathcal{B}|$.*

We now know, from Theorem 3.41, that all the family of projective planes $PG(2, q)$ are excluded minors for *wHPP*-representability. Is this the complete set of excluded minors?.

Question 3.46. *What are the excluded minors for wHPP-representability?.*

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