GMT seminar 5

1 The structure theorem

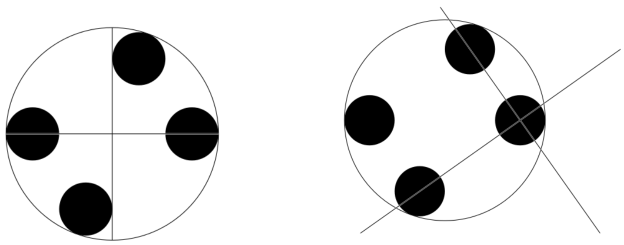
A (σ-finite) purely n-unrectifiable set in $\mathbb{R}^{n+k}$ has H^n-measure zero projection onto almost every plane.

1.1 Purely unrectifiable sets

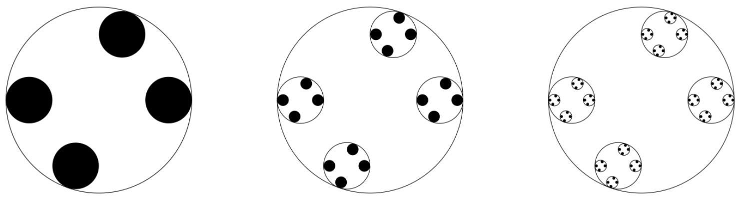
A set is *purely n-unrectifiable* if it contains no countably n-rectifiable subsets of positive H^n-measure.

1.1.1 Example from [1]

Consider the set P_1 consisting of four disks of radius 1/4, and consider the lines ℓ_1 and ℓ_2:



Iterate the construction to form $P_1 \supset P_2 \supset \ldots \supset P$:



Projections Π_+ and Π_- on these lines yield measure zero sets.
Let $\gamma\colon [0,1] \to \mathbb{R}$ be a Lipschitz curve.
Let $A = \gamma^{-1}(P)$. Then $H^1(\Pi_+(\gamma(A))) = 0$ so $f'(t) \perp \ell_+$ for a.e. $t \in A$.

Similarly $f'(t) \perp \ell_-$ for a.e. $t \in A$. Hence A has measure zero so $f(A) = \operatorname{im} f \cap P$ has measure zero. Hence P is purely unrectifiable.

Note that P projects both horizontally and vertcally onto the interval $[0, 1]$.

1.2 Idea for a proof by induction

A proof by induction (based on Besicovitch's theorem) due to Brian White can be found in [2].

Construction: Consider a set $X \subset \mathbb{R}^{n+k}$ with nonzero H^k measure and with positive projection onto a positive-measure set of k-planes. Define $f\colon \mathbb{R}^k \to \mathbb{R}^{n+k}$ by

$$f(x) = \min\{y \in \mathbb{R}^n \colon (x, y) \in X\}$$

where $\mathbb{R}^n$ is ordered lexicographically. It turns out that this function (or a similarly defined function after rotating $\mathbb{R}^{n+k}$) is approximately differentiable in a sense such that it restricts to a Lipschitz function on some set of positive measure. Its graph then has positive-measure intersection with X.

Example: With $n + k = 3$ and $k = 2$, this gives the "lower boundary" of the set. With $n + k = 3$ and $k = 1$ it gives the lower-left point in each slice.

To prove that f is approximately differentiable one uses the structure theorem for curves in $\mathbb{R}^{n+1}$, which can be established using induction.

2 Sets of locally finite perimeter

Remarkable that the simplest intuition for tangent space works for such general sets: Zoom in until the set becomes a half-space.

2.1 De Giorgi's theorem

2.1.1 Idea of statement

Let E have locally finite perimeter. Then E is infinitesimally a half-space with inward unit normal

$$\nu_E(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} \nu \, d\mu_E}{\mu_E(B_\rho(x))}.$$

2.1.2 Idea of proof

Proof idea: Pick a point on $\partial^* E$ and zoom in, showing that the result is the claimed half-space.

Two alternate views:

1. Zoom in by looking at $E \cap B_\rho$ for smaller and smaller ρ.
2. Blow up E by considering $\rho^{-1}E$ for smaller and smaller ρ.

They are equivalent (up to a coordinate change), but give different limits.

First part: Derive the bound (6) of the form

$$\mu_E(B_\rho) \leq C\rho^n.$$

By a coordinate change this gives

$$\mu_{\rho^{-1}E}(B_1) \leq C$$

allowing the compactness theorem for BV functions to be used to get a limit set F.

Second part, cute trick: Rewrite derivative $\int (D_i\zeta) dL^{n/1}$ as $\int \zeta \nu_i d\mu_E$, which goes to zero when zooming in if $i \neq n+1$.

$$\int_{\mathbb{R}^{n+1}} \chi_F D_i \zeta \, dL^{n+1} = 0$$

so

$$\int_{\mathbb{R}} \chi_F D_i \zeta \, dL^{n+1} \quad \text{for almost every } y \in \mathbb{R}^n$$

so

$$\int_a^b \chi_F(x,y) dx \quad \text{is independent of } a \text{ and } b$$

so $\chi_F \, dx$ is translation-invariant. Hence $\chi_F(x,y) = 1$ or $\chi_F(x,y) = 0$ for a.e. x.
Hence $F = \mathbb{R}^n \times H$ for some $H \subseteq \mathbb{R}$.

The same trick in direction $n+1$ gives an inequality yielding $H = (\lambda, \infty)$ for some $\lambda \in [-\infty, \infty]$.

Third part: Using the Sobolev inequality to show that

$$L^{n+1}(B_\rho \cap E) \geq c\rho^{n+1}$$

and

$$L^{n+1}(B_\rho \setminus E) \geq c\rho^{n+1}$$

for all small ρ.
Hence $\lambda = 0$.

2.1.3 Why precisely locally finite perimeter?

- The compactness theorem for BV_{loc} functions.
- Allows expressing derivatives $\int (D_i\zeta) dL^{n+1}$ as $\int \zeta \nu_i d\mu_E$.

References

[1] Harold R. Parks. Purely unrectifiable sets with large projections. *Aust. J. Math. Anal. Appl.*, 6(1):Art. 17, 10, 2009.

[2] Brian White. A new proof of Federer's structure theorem for k-dimensional subsets of $\mathbf{R}^N$. *J. Amer. Math. Soc.*, 11(3):693–701, 1998.