An Example related to Seminar 5.

During the seminar I tried to give an example relating to the final part of Theorem 14.2. In particular I wanted to indicate that the Sobolev inequality part of the argument was quite subtle and that it was not obvious at all that $\lambda = 0$. I thought that I should write down the example with some care.

The statement: *Let $u \in BV$ be a function and $\nabla u = \sigma(x)\mu$ for some function $|\sigma(x)| = 1$ and measure μ. Assume furthermore that $0 \in spt(\mu)$ and that*

$$\lim_{\rho \to 0} \frac{\int_{B_\rho(0)} \nabla u(x) dx}{\mu(B_\rho(0))} = -e_{n+1} \tag{1}$$

then it doesn't necessary follow that $\mu_0 = \lim_{\rho \to 0} \frac{\mu(\rho \cdot)}{\mu(B_\rho(0))}$ satisfies $spt(\mu_0) = \{x_{n+1} = 0\}$. We may even choose u so that $spt(\mu)$ has zero $(n+1)-$dimensional measure.

Clearly this shows that the final part of the argument of Theorem 14.2 depends on the assumption that we have a set of finite perimeter (and not just any BV function.)

The Example: *We define $u(x) \in BV(-1,1)$ (the one dimensional case) as*

$$u(x) = \begin{cases} \sum_{k=1}^{n} \frac{1}{(k!)^2} & \text{if } x \in \left(\frac{1}{(n+1)!}, \frac{1}{n!} \right] \\ \sum_{k=1}^{\infty} \frac{1}{(k!)^2} & \text{if } x \leq 0. \end{cases}$$

Then it follows that

$$u'(x) = -\sum_{n=1}^{\infty} \frac{1}{((n+1)!)^2} \delta_{1/n!}$$

where δ is the dirac delta measure.

Clearly (1) holds since $\sigma(x) = -1$ on the support of $u'(x)$. Moreover, if we make the blow-up with $\rho_m = \frac{2}{m!}$ then, on the set $(-1,1)$,

$$\mu_0 = \lim_{\rho_m \to 0} \frac{\mu(\rho_m \cdot)}{\mu(B_{\rho_m}(0))} = \frac{\sum_{n=m}^{\infty} \frac{1}{((n+1)!)^2} \delta_{m!/(2(n!))}}{\sum_{n=m}^{\infty} \frac{1}{((n+1)!)^2}}. \tag{2}$$

But it is only the first term in the numerator that will affect the limit since if we consider the numerator divided my the first term we get

$$\frac{\sum_{n=m}^{\infty} \frac{1}{((n+1)!)^2}}{\frac{1}{((m+1)!)^2}} = \sum_{n=m}^{\infty} \frac{((m+1)!)^2}{((n+1)!)^2} \leq 1 + \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^2} \to 1, \tag{3}$$

as $m \to \infty$.

Using (3) in (2) we can deduce that

$$\mu_0 = \delta_{1/2}.$$

By choosing $\rho_k = \frac{s}{m!}$, for $s > 1$, we can assure that $\mu_0 = \delta_{1/s}$ - that is the support of μ_0 will depend on the particular subsequence we choose and the support of μ_0 may not be the origin.