GMT Seminar 15th January.

The most important Theorem in this weeks reading is Theorem 14.3. It is a very profound and deep result. It is worthwhile to focus a little extra on the result ant on its proof. I have the following questions relating to the Theorem:

Question 1: What is the geometric meaning of $\partial^* E$ defined in formula (14.2). More specifically, what is the meaning of the limit

$$\lim_{\rho \to 0} \frac{\int_{B_\rho(x)} \nu d\mu_E}{\mu_E(B_\rho(x))} \text{ exists and has length 1?} \tag{1}$$

Question 2: Are there any counterexamples to the existence of the limit in (1)?

Question 3: Are there any examples where the limit (1) exists but has non-unit length?

Question 4: I mentioned above that Theorem 14.3 is deep. Why is it so deep?[1]

Question 5: Why does the right hand side in equation (4) on p.74 exist for a.e. ρ?

Question 6: What does equation (5) on page 74 mean geometrically?

Question 7: Can you say something about the first equation, ant the line following that equation, on page 75? That is the pivotal point of the proof. [2]

Question 8: How does it follow that χ_H is non-decreasing in $\mathbb{R}$ on p.75?

Question 9: After equation (8) on p.75 follows a rather difficult (at least non-intuitive) argument to show that $\lambda = 0$.

1. Is it, or isn't it, obvious that the tangent plane must cut through the origin since we assume that $0 = y \in \text{spt}(\mu_E)$?

2. Can one find an example of a BV function[3] $u(x)$ such that if $\nabla u(x) = \nu(x)\mu(x)$ for some vector $\nu(x)$ with $|\nu(x)| = 1$ a.e. and μ a measure and $0 \in \text{spt}(\mu)$ and

$$\lim_{\rho \to 0} \frac{\int_{B_\rho(0)} \nu(x)\mu(x)}{\mu(B_\rho(0))} = (0, 0, \ldots, 0, 1)$$

[1] Of course, I may be mistaken and it might be trivial - in that case why?

[2] This question is very undefined - but it is worth focusing on that equation for a few minutes.

[3] Obviously not a function of the form χ_E.

but, for some subsequence $\rho_j \to 0$,

$$\operatorname{spt}\left(\lim_{\rho_j \to 0} \frac{\nabla u_j(\rho x)}{\mu(B_{\rho_j}(0))}\right) \neq \{y \in \mathbb{R}^n;\ y_{n+1} = 0\}?$$

In that case, how is it used that the BV function is of the form χ_E in the proof of Theorem 14.3?

Question 10: Exactly what is integrated on p. 76 to deduce equation (9)?