GMT Seminar 29th January.

Question 1: *[Courtesy of Eric Larsson]*

1. In Lemma 17.11, what is the interpretation of $H \in L^p_{loc}(\mu)$?
2. Is there a good example where this property fails?
3. What is the geometric interpretation of the conclusion of Lemma 17.11?

Question 2: *[Courtesy of Andreaas Minne]*

1. What is the geometric interpretation of 17.3 and/or of one of its corollaries: 17.4 and 17.5?
2. Do you have any cool examples of stationary varifolds?

Question 3: What is the relation between Definition 16.3 and 16.4. In particular, why is the defining formula in Definition 16.3

$$\int \mathrm{div}_M(X) d\mu_V = 0,$$

with zero right hand side. And the defining formula in Definition 16.4

$$\int \mathrm{div}_M(X) d\mu_V = -\int X \cdot \overline{H}_M d\mu_V$$

with non-zero right hand side.

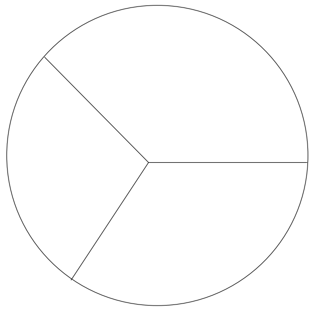
Question 4: A stationary varifold is "supposed to be a minimal surface" but I quite frankly do not understand the presence of θ in the definition. I thought that we should discuss that and see if we can make any sense of the multiplicity function in the definition.

1. Let $V = (M, \theta)$ be a varifold given by a rectifiable set $M = \{x \in \mathbb{R}^2;\ x_2 = 0 \text{ and } x_1 \in [-1, 1]\}$ and a density function

   $$\theta(x) = \begin{cases} 1 & \text{if } x_1 \leq 0 \\ 2 & \text{if } x_1 > 0. \end{cases}$$

   Is V stationary?

2. Consider the propeller varifold in the picture below (the varifold is the three straight lines - the circle is just there to remind us of peace, love and understanding). All the angles are $2\pi/3$.



Is the varifold stationary if $\theta = 1$? What if $\theta(x) = \begin{cases} 1 & \text{if } x_1 \leq 0 \\ 2 & \text{if } x_1 > 0? \end{cases}$

Question 5: In Corollary 17.8, page 86, it is claimed that $\theta^n(\mu, x)$ is upper semi-continuous. But if we assume that $V = (M, \theta)$ is a stationary varifold with multiplicity function $\theta = 1$, so that $\mu = \mathcal{H}^n\big|_M$. Will it follow that $\theta^n(\mu, x)$ is continuous?

Question 6: *(More like a homework, and it involves some PDE theory.)* A minimal graph over some open set $U \subset \mathbb{R}^n$ is the graph of some function $f \in C^{0,1}(U)$ such that the f minimizes the following energy among all functions with the same boundary data

$$\int_U \sqrt{1 + |\nabla f(x)|^2} dx.$$

1. What is the statement of the monotonicity formula for a minimal graph?
2. Use the monotonicity formula to prove that for almost every $x^0 \in U$, such that $B_R(x^0) \subset U$ for some $R > 0$, the following estimate holds

$$\sqrt{1 + |\nabla f(x^0)|^2} \leq \frac{1}{\omega_n R^n} \int_{B_R(x^0)} \sqrt{1 + |\nabla f(x)|^2} dx,$$

   that is the integral of the gradient controls the point-wise value of the gradient.
3. For a harmonic function $u(x)$, that is $\Delta u(x) = 0$, it follows that $\Delta |\nabla u(x)|^2 = 2 \sum_{i,j=1}^n |\partial_{ij} u(x)|^2 \geq 0$. That is, the gradient squared is sub-harmonic so it satisfies the sub-meanvalue property. Is there any relation to the monotonicity formula?