GMT Seminar 16th February.

Question 1. Suppose that $k = n = 1$ and that V is a line segment orthogonal to $\mathbb{R}^n$. Then there can be no good approximation of V as a (Lipschitz) graph over $\mathbb{R}^n$. How does this example relate to Theorem 20.2?

Question 2. Did anyone make sense of the reference to the Cauchy inequality in deducing the fourth displayed formula from the third on p. 104?

Question 3. What is the reasoning in line 7, when choosing a "suitable" ϕ in 17.2?

Question 4. Let $M = \{(x', f(x'));\ x' \in B_1'(0)\}$, where $x' = (x_1, x_2, ..., x_{n-1})$ and $B_1'(0) = \{x';\ |x'| < 1\}$. Is there a more common way to write

$$\int_{B_1'(0)} \mathrm{dist}((x', f(x')), \{x_n = 0\})^2 \mathcal{H}^{n-1}(x')?$$

If so why does Simon use the $\mathrm{dist}(x - \xi, T)^2$ notation in Lemma 22.2?

Question 5. Assume that the multiplicity $\theta(x) = 1$ for every $x \in M$ where $V = (M, \theta)$ is a stationary n-varifold in $\mathbb{R}^{n+1}$. Assume also that $\{x_{n+1} = 0\}$ is an approximate tangentplane at $0 \in V$:

$$\lim_{\lambda \to 0^+} \int_{B_1(0) \cap \{y;\ \lambda y \in M\}} f(y) d\mathcal{H}^n(y) = \int_{B_1(0) \cap \{x_{n+1} = 0\}} f(y) d\mathcal{H}^n(y)$$

for all $f \in C_c(B_1(0))$.

Does it follow that

$$\lim_{\lambda \to 0^+} \int_{B_1(0)} \mathrm{dist}(y, \{y_{n+1} = 0\})^2 \mathcal{H}^n \lfloor_M (y) = 0?$$

Question 6.

1. How can one argue (informally) that the graph over $B_1(0)$ with minimal area and boundary values $g(x) \in C^{0,1}(\partial B_1(0))$ minimizes the energy

$$J(f) = \int_{B_1(0)} \sqrt{1 + |\nabla f(x)|^2} dx \tag{1}$$

   among all functions such that $f(x) = g(x)$ on $\partial B_1(0)$?

2. How can one deduce that if $f(x)$ minimizes the functional $J(f)$ then

$$\int_{B_1(0)} \frac{\nabla \phi(x) \cdot \nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}} dx = 0 \tag{2}$$

   for all $\phi \in C_c^1(B_1(0))$.

3. Choosing $\phi(x) = \psi^2(x) f(x)$ in (2) where

$$\psi(x) = \begin{cases} 1 & \text{when } |x| \leq 1/2 \\ 0 & \text{when } |x| > 3/4, \end{cases}$$

and $|\nabla\psi(x)| \le 8$. How does this lead to

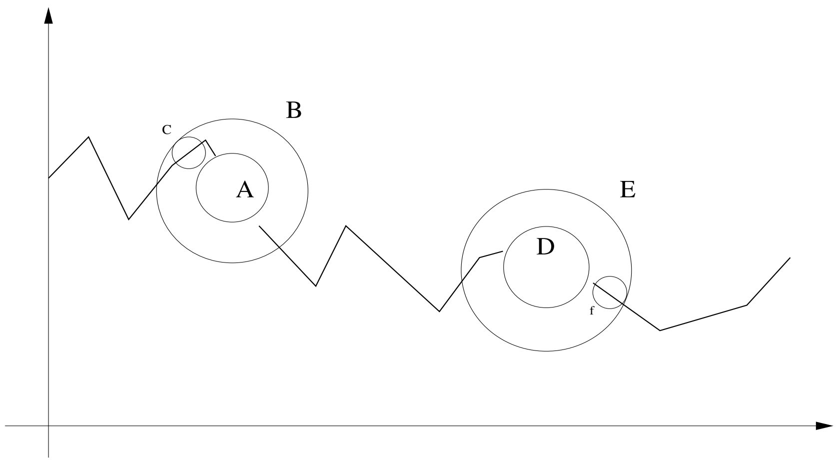
$$\int_{B_{1/2}(0)} \frac{|\nabla f(x)|^2}{\sqrt{1+|\nabla f(x)|^2}} dx \le C \int_{B_1(0)} |f(x)|^2 dx,$$

and how does this relate to Lemma 22.2?

Question 7. Given $\epsilon, \kappa > 0$ and assume that $\int_{B_1(0)} |\nabla f(x)|^2 dx \le \delta$.

1. What is the maximal measure that the set where $|\nabla f(x)| > \kappa$ can have?
2. Let $D \subset B_1(0)$ and assume that we can cover D by balls $B_{\rho_j}(x^j)$ such that
$$\frac{1}{|B_{\rho_j}(x^j)|} \int_{B_{\rho_j}(x^j} |\nabla f(x)|^2 dx \ge \kappa.$$
What is the maximal measure of D?
3. How does this relate to the proof, in Theorem 20.2. that
$$\mathcal{H}^n(\mathrm{spt}(V) \setminus \mathrm{graph}(f)) \le cl^{-2n-2}E?$$

Question 8. Below you can find a picture of a graph of a stationary varifold V with some holes in it. We put two balls A and D in the holes and make sure that the concentric balls with twice the radius, called B and E, intersects the graph of f in at the center of two balls, c and f, such that $c \subset B \setminus A$ and $f \subset D \setminus E$ and c, f have radius ρ_c, ρ_f comparable to the radius ρ_A, ρ_D of A and D.



1. How does the monotonicity formula imply that $\mu(B)/\rho_B^n \approx 1$?
2. What is the value of $\mu(A)$?
3. Given that the monotonicity functional states that
$$\frac{\mu(B)}{\rho_B^n} - \frac{\mu(A)}{\rho_A^n} = \int_{B\setminus A} \underbrace{\frac{|D^{\perp} r|^2}{r^n}}_{\approx \frac{|p_x - p|^2}{\rho_A^n}} d\mu$$

can we conclude that having a "hole" of radius ρ_A in the varifold V has the "cost" $\int_B |p_x - p|^2 d\mu \approx \rho_A^n$. Does that say anything about how large the measure of the "holes" can be if $\int_{B_1(0)} |p_x - p|^2 d\mu \leq \delta$?

4. Relate this to the estimate, in the proof of Theorem 20.2

$$\mathcal{H}^n(\mathrm{graph}(f) \setminus \mathrm{spt}(V)) \leq c l^{-2n-2} E?$$

Question 9. Let $V = (M, \theta)$, $\theta = 1$, be a stationary n-varifold in $\mathbb{R}^{n+1}$ and assume that $f(x)$ is a Lipschitz function such that

$$\mathcal{H}^n(\mathrm{graph}(f) \setminus \mathrm{spt}(V)) + \mathcal{H}^n(\mathrm{spt}(V) \setminus \mathrm{graph}(f)) \leq c l^{-2n-2} E. \tag{3}$$

1. Since V is stationary it follows that $\int_{B_1(0)} \mathrm{div}_M(X) d\mu = 0$. Can you use this to make sense of the following informal calculation, together with half of (3), with $X = \phi(x) e_{n+1}$

$$0 = \int_{B_1} \mathrm{div}_M(X) d\mu \approx \underbrace{\int_{B_1'(x)} \frac{\nabla\phi \cdot \nabla f}{\sqrt{1 + |\nabla f|^2}} dx'}_{x' = (x_1, x_2, \ldots, x_n)} \approx \left\{ \begin{array}{l} \text{if} \\ |\nabla f| << 1 \end{array} \right\} \approx$$

$$\approx \int_{B_1'(x)} \nabla\phi \cdot \nabla f dx',$$

thus f satisfies the conditions in the harmonic approximation lemma.

2. Can you use the other half of (3) to conclude that

$$\int_{B_1(0)} \mathrm{dist}(x, \{x_{n+1} = 0\})^2 d\mu \approx \int_{B_1'(0)} |f(x)|^2 dx.$$

Does this mean that you can transfer the information from the harmonic approximation lemma for f to information about V?

Question 10. Can you try to summarize the entire construction in this weeks reading. What are the main ideas?

[REMARK: *It is well known that the statement in Theorem 22.5 implies that M is a $C^{1,\alpha}$ manifold. Something we will see next week.*]